Chapter 1
Introduction

1.1 Definition:

Definition of Stress

Consider a small area δA on the surface of a body (Fig. 1.1). The force acting on this area is δF This force can be resolved into **two perpendicular components**

- The component of force acting normal to the area called **normal** force and is denoted by δF_n
- The component of force acting along the plane of area is called **tangential** force and is denoted by δF_t

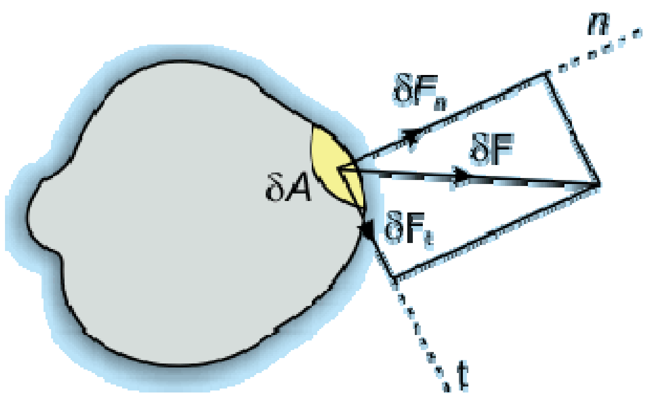

Fig 1.1 Normal and Tangential Forces on a surface

When they are expressed as force per unit area they are called as **normal stress** and **tangential stress** respectively**.** The tangential stress is also called shear stress

The normal stress

$$\sigma = \lim_{\delta A \to 0}\left(\frac{\delta F_n}{\delta A}\right) \qquad (1.1)$$

And shear stress

$$\tau = \lim_{\delta A \to 0}\left(\frac{\delta F_t}{\delta A}\right) \qquad (1.2)$$

Definition of Fluid

- A fluid is a substance that **deforms continuously** in the face of tangential or shear stress, **irrespective of the magnitude of shear stress** .This continuous deformation under the application of shear stress constitutes a flow.

- In this connection fluid can also be defined as the **state of matter that cannot sustain any shear stress.**

Example : Consider Fig 1.2

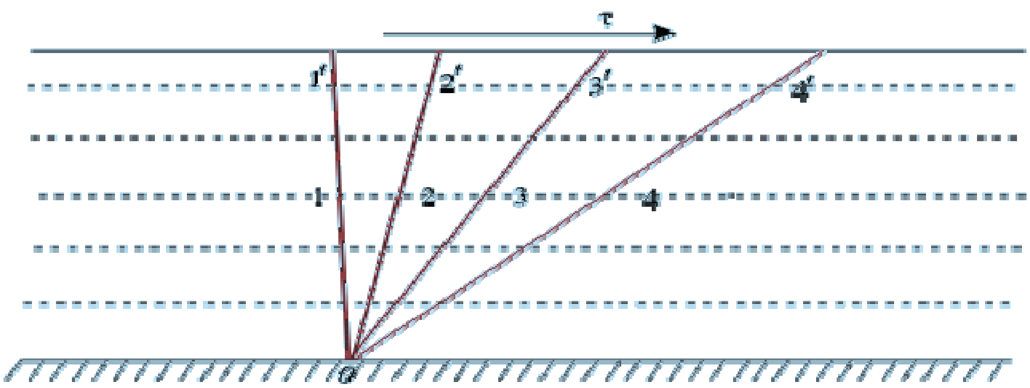

Fig 1.2 Shear stress on a fluid body

If a shear stress τ is applied at any location in a fluid, the element 011' which is initially at rest, will move to 022', then to 033'. Further, it moves to 044' and continues to move in a similar fashion.

In other words, the **tangential stress in a fluid body depends on velocity of deformation and vanishes as this velocity approaches zero. A good example is Newton's parallel plate experiment where dependence of shear force on the velocity of deformation was established.**

1.2 Science of Fluid Mechanics:

Distinction Between Solid and Fluid

<table>
<tr><th>Solid</th><th>Fluid</th></tr>
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- More Compact Structure
- Attractive Forces between the molecules are larger therefore more closely packed
- Solids can resist tangential stresses in static condition
- Whenever a solid is subjected to shear stress
  - a. It undergoes a definite deformation α or breaks
  - b. α is proportional to shear stress upto some limiting condition
- Solid may regain partly or fully its original shape when the tangential stress is removed

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- Less Compact Structure
- Attractive Forces between the molecules are smaller therefore more loosely packed
- Fluids cannot resist tangential stresses in static condition.
- Whenever a fluid is subjected to shear stress
  - a. No fixed deformation
  - b. Continious deformation takes place until the shear stress is applied
- A fluid can never regain its original shape, once it has been distorded by the shear stress

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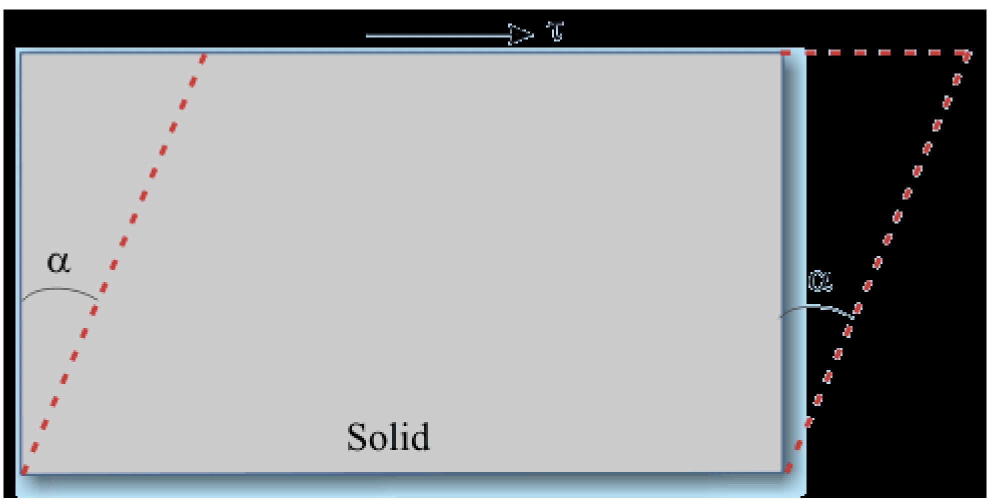
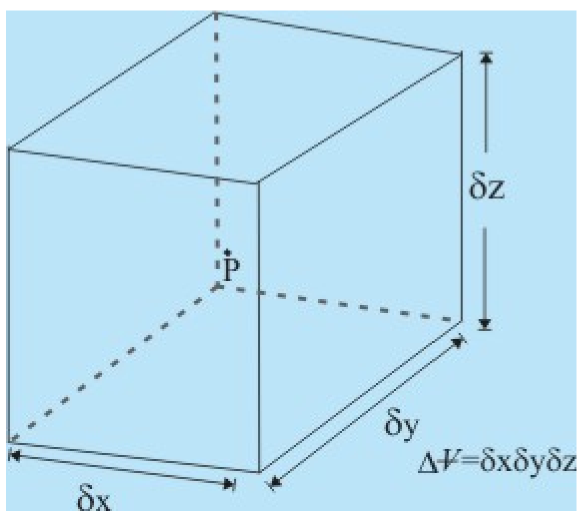


Fig 1.3 Deformation of a Solid Body

1.3 Fluid Properties:

Characteristics of a continuous fluid which are independent of the motion of the fluid are called basic properties of the fluid. Some of the basic properties are as discussed below.

Property	Symbol	Definition	Unit
Density	ρ	The density *p* of a fluid is its mass per unit volume . If a fluid element enclosing a point *P* has a volume Δ∀ and mass Δm (Fig. 1.4), then density (ρ)at point *P* is written as $\rho = \lim_{\Delta\forall \to \Delta\forall_0} \left(\frac{m}{\Delta\forall}\right)$ However, in a medium where continuum model is valid one can write - $\rho = \lim_{\Delta\forall \to 0} \left(\frac{m}{\Delta\forall}\right) = \left[\frac{dm}{d\forall}\right]_{\forall}$ (1.3)  Fig 1.4 A fluid element enclosing point P	kg/m^3
Specific Weight	γ	The **specific weight** is the weight of fluid per unit volume. The specific weight is given	N/m^3

		by $\gamma = \rho g$ (1.4) Where *g* is the gravitational acceleration. Just as weight must be clearly distinguished from mass, so must the specific weight be distinguished from density.	
Specific Volume	v	The **specific volume** of a fluid is the volume occupied by unit mass of fluid. Thus $v = \frac{1}{\rho}$ (1.5)	m^3 /kg
Specific Gravity	s	For liquids, it is the ratio of density of a liquid at actual conditions to the density of pure water at 101 kN/m^2 , and at 4°C. The specific gravity of a gas is the ratio of its density to that of either hydrogen or air at some specified temperature or pressure. **However, there is no general standard; so the conditions must be stated while referring to the specific gravity of a gas.**	-

Viscosity (μ) :

- Viscosity is a fluid property whose effect is understood when the fluid is in motion.
- In a flow of fluid, when the fluid elements move with different velocities, each element will feel some resistance due to fluid friction within the elements.
- Therefore, shear stresses can be identified between the fluid elements with different velocities.
- The relationship between the shear stress and the velocity field was given by Sir Isaac Newton.

Consider a flow (Fig. 1.5) in which all fluid particles are moving in the same direction in such a way that the fluid layers move parallel with different velocities.

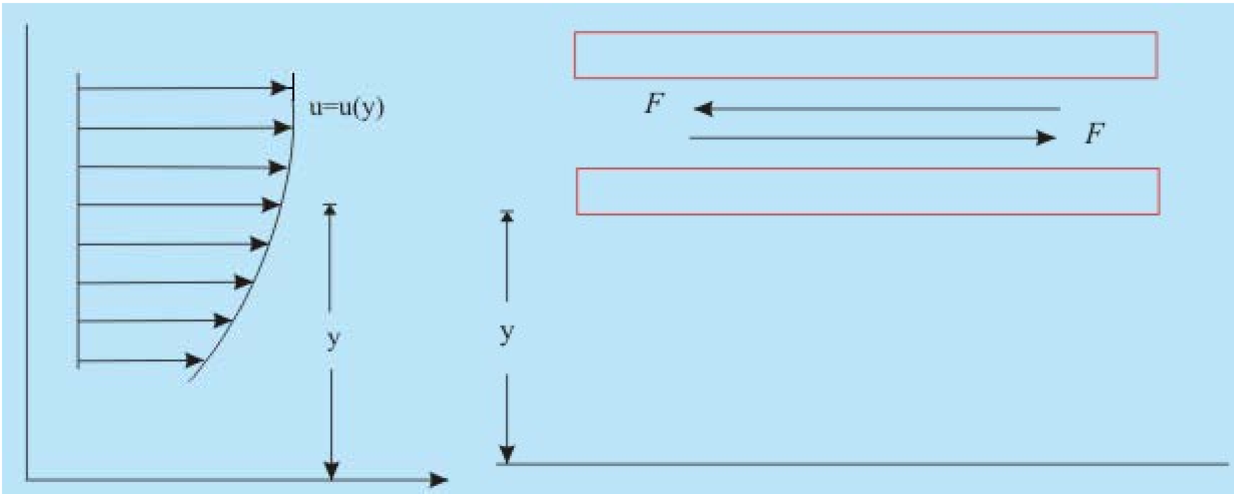


Fig 1.5 Parallel flow of a fluid Fig 1.6 Two adjacent layers of a moving fluid.

- The upper layer, which is moving faster, tries to draw the lower slowly moving layer along with it by means of a force *F* along the direction of flow on this layer. Similarly, the lower layer tries to retard the upper one, according to Newton's third law, with an equal and opposite force *F* on it (Figure 1.6).
- Such a fluid flow where x-direction velocities, for example, change with y-coordinate is called **shear flow** of the fluid.
- Thus, the dragging effect of one layer on the other is experienced by a tangential force *F* on the respective layers. If *F* acts over an area of contact *A,* then the shear stress τ is defined as

$$\tau = F/A$$

- Newton postulated that τ is proportional to the quantity $\Delta u/ \Delta y$ where Δy is the distance of separation of the two layers and Δu is the difference in their velocities.
- In the limiting case of $\Delta y \rightarrow 0$, $\Delta u / \Delta y$ equals du/dy, the velocity gradient at a point in a direction perpendicular to the direction of the motion of the layer.
- According to Newton τ and du/dy bears the relation

$$\tau = \mu \frac{du}{dy} \qquad (1.7)$$

where, the constant of proportionality μ is known as the **coefficient of viscosity** or simply viscosity which is a property of the fluid and depends on its state. Sign of τ depends upon the sign of du/dy. For the profile shown in Fig. 1.5, du/dy is positive everywhere and hence, τ is positive. Both the velocity and stress are considered positive in the positive direction of the

coordinate parallel to them.
Equation

$$\tau = \mu \frac{du}{dy}$$

defining the viscosity of a fluid, is known as Newton's law of viscosity. Common fluids, viz. water, air, mercury obey Newton's law of viscosity and are known as *Newtonian fluids.*

Other classes of fluids, viz. paints, different polymer solution, blood do not obey the typical linear relationship, of τ and du/dy and are known as **non-Newtonian fluids**. In non-newtonian fluids viscosity itself may be a function of deformation rate as you will study in the next lecture.

Causes of Viscosity

- The causes of viscosity in a fluid are possibly attributed to two factors:

(i) intermolecular force of cohesion
(ii) molecular momentum exchange

- Due to strong cohesive forces between the molecules, any layer in a moving fluid tries to drag the adjacent layer to move with an equal speed and thus produces the effect of viscosity as discussed earlier. Since cohesion decreases with temperature, the liquid viscosity does likewise.

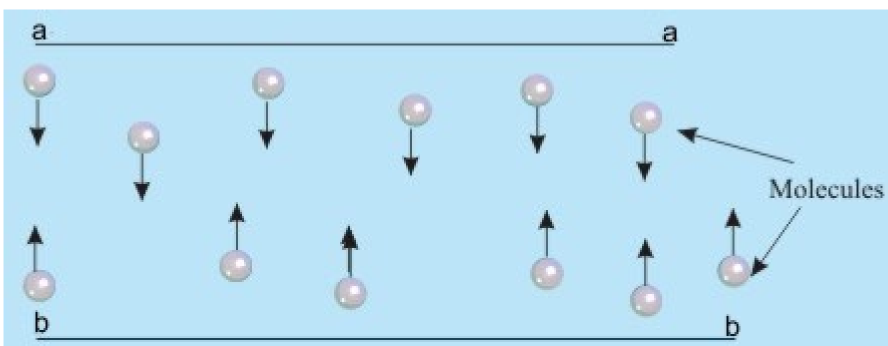


Fig 1.7 Movement of fluid molecules between two adjacent moving layers

- Molecules from layer aa in course of continous thermal agitation migrate into layer bb
- Momentum from the migrant molecules from layer aa is stored by molecules of layer bb by way of collision
- Thus layer bb as a whole is speeded up
- Molecules from the lower layer bb arrive at aa and tend to retard the layer aa

- Every such migration of molecules causes forces of acceleration or deceleration to drag the layers so as to oppose the differences in velocity between the layers and produce the effect of viscosity.

- As the random molecular motion increases with a rise in temperature, the viscosity also increases accordingly. Except for very special cases (e.g., at very high pressure) the viscosity of both liquids and gases ceases to be a function of pressure.

- For Newtonian fluids, the coefficient of viscosity depends strongly on temperature but varies very little with pressure.

- For liquids, molecular motion is less significant than the forces of cohesion, thus **viscosity of liquids decrease with increase in temperature.**

- For gases,molecular motion is more significant than the cohesive forces, thus **viscosity of gases increase with increase in temperature.**

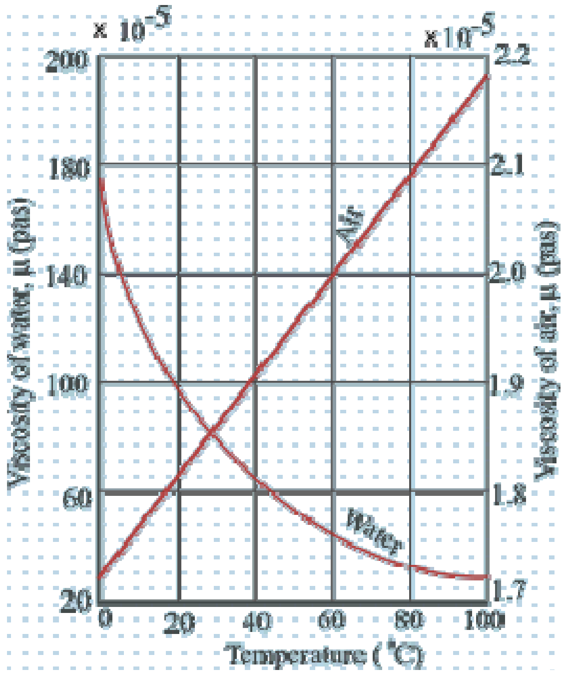


Fig 1.8: Change of Viscosity of Water and Air under 1 atm

No-slip Condition of Viscous Fluids

- It has been established through experimental observations that the relative velocity between the solid surface and the adjacent fluid particles is zero whenever a viscous fluid flows over a solid surface. This is known as no-slip condition.

- This behavior of no-slip at the solid surface is not same as the wetting of surfaces by the fluids. For example, mercury flowing in a stationary glass tube will not wet the surface, but will have zero velocity at the wall of the tube.

- The wetting property results from surface tension, whereas the no-slip condition is a consequence of fluid viscosity.

1.4 Capillarity:

- The interplay of the forces of cohesion and adhesion explains the phenomenon of capillarity. When a liquid is in contact with a solid, if the forces of adhesion between the molecules of the liquid and the solid are greater than the forces of cohesion among the liquid molecules themselves, the liquid molecules crowd towards the solid surface. The area of contact between the liquid and solid increases and the liquid thus wets the solid surface.

- The reverse phenomenon takes place when the force of cohesion is greater than the force of adhesion. These adhesion and cohesion properties result in the phenomenon of capillarity by which a liquid either rises or falls in a tube dipped into the liquid depending upon whether the force of adhesion is more than that of cohesion or not (Fig.2.4).

- The angle θ as shown in Fig. 2.4, is the area wetting contact angle made by the interface with the solid surface.

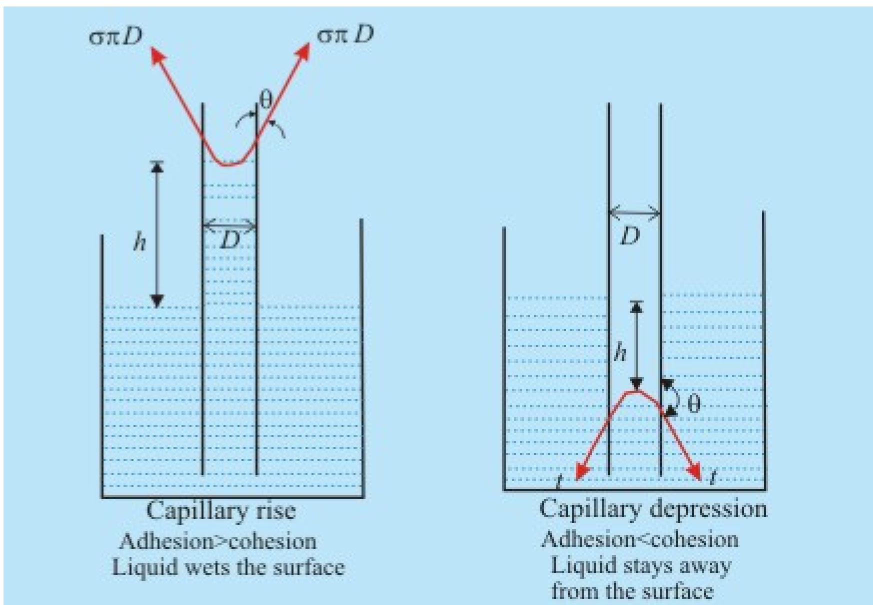


Fig 2.4 Phenomenon of Capillarity

- For pure water in contact with air in a clean glass tube, the capillary rise takes place with θ = 0 . Mercury causes capillary depression with an angle of contact of about 130^0 in a clean glass in contact with air. Since h varies inversely with D as found from Eq. ($h = \frac{4\sigma\cos\theta}{\rho g D}$), an appreciable capillary rise or depression is observed in tubes of small diameter only.

1.5 Surface Tension:

- The phenomenon of surface tension arises due to the two kinds of intermolecular forces

  (i) **Cohesion :** The force of attraction between the molecules of a liquid by virtue of which they are bound to each other to remain as one assemblage of particles is known as the force of cohesion. This property enables the liquid to resist tensile stress.

  (ii) **Adhesion :** The force of attraction between unlike molecules, i.e. between the molecules of different liquids or between the molecules of a liquid and those of a solid body when they are in contact with each other, is known as the force of adhesion. This force enables two different liquids to adhere to each other or a liquid to adhere to a solid body or surface.

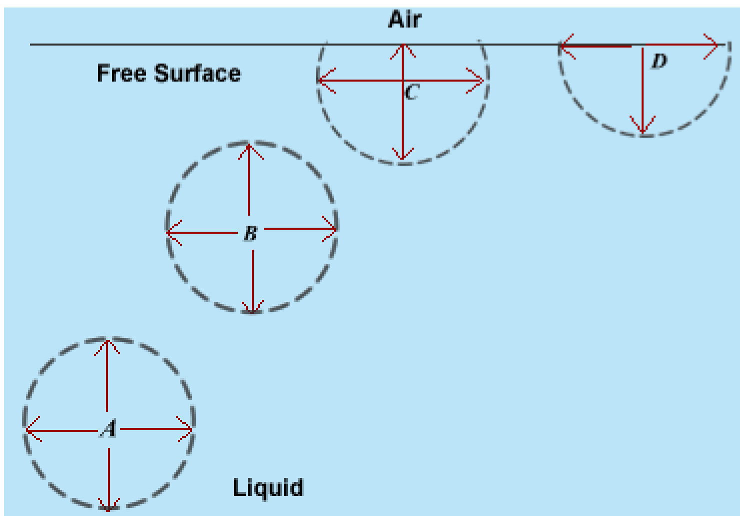


Figure 2.3 The intermolecular cohesive force field in a bulk of liquid with a free surface

A and B experience equal force of cohesion in all directions, C experiences a net force interior of the liquid The net force is maximum for D since it is at surface

- Work is done on each molecule arriving at surface against the action of an inward force. Thus mechanical work is performed in creating a free surface or in increasing the area of the surface. Therefore, a surface requires mechanical energy for its formation and the existence of a free surface implies the presence of stored mechanical energy known as free surface energy. Any system tries to attain the condition of stable equilibrium with its potential energy as minimum. Thus a quantity of liquid will adjust its shape until its surface area and consequently its free surface energy is a minimum.

- The magnitude of surface tension is defined as the tensile force acting across imaginary short and straight elemental line divided by the length of the line.

- The dimensional formula is F/L or MT^{-2} . It is usually expressed in N/m in SI units.

- Surface tension is a binary property of the liquid and gas or two liquids which are in contact with each other and defines the interface. It decreases slightly with increasing temperature. The surface tension of water in contact with air at 20°C is about 0.073 N/m.

- It is due to surface tension that a curved liquid interface in equilibrium results in a greater pressure at the concave side of the surface than that at its convex side.

1.6 Compressibility:

- Compressibility of any substance is the measure of its change in volume under the action of external forces.
- The normal compressive stress on any fluid element at rest is known as hydrostatic pressure p and arises as a result of innumerable molecular collisions in the entire fluid.

- The degree of compressibility of a substance is characterized by the bulk modulus of elasticity E defined as

$$E = \lim_{\Delta\forall \to 0}\left(\frac{-\Delta p}{\Delta\forall / \forall}\right) \qquad (2.3)$$

- Where $\Delta\forall$ and Δp are the changes in the volume and pressure respectively, and $\forall$is the initial volume. The negative sign (-sign) is included to make E positive, since increase in pressure would decrease the volume i.e for $\Delta p>0$, $\Delta\forall<0$) in volume.
- For a given mass of a substance, the change in its volume and density satisfies the relation

Dm = 0, $D(\rho\forall) = 0$

$$\frac{\Delta\forall}{\forall} = -\frac{\Delta\rho}{\rho} \qquad (2.4)$$

using $E = \lim_{\Delta\forall \to 0}\left(\frac{-\Delta p}{\Delta\forall / \forall}\right)$ & $\frac{\Delta\forall}{\forall} = -\frac{\Delta\rho}{\rho}$

we get

$$E = \lim_{\Delta\rho \to 0}\left(\frac{\Delta p}{\Delta\rho / \rho}\right) = \rho\frac{dp}{d\rho} \qquad (2.5)$$

- Values of *E* for liquids are very high as compared with those of gases (except at very high pressures). Therefore, liquids are usually termed as incompressible fluids though, in fact, no substance is theoretically incompressible with a value of *E* as ∞.

- For example, the bulk modulus of elasticity for water and air at atmospheric pressure are approximately 2 x 10^6 kN/m ^{2} and 101 kN/m ^{2} respectively. It indicates that air is about 20,000 times more compressible than water. Hence water can be treated as incompressible.

- For gases another characteristic parameter, known as compressibility *K,* is usually defined , it is the reciprocal of *E*

$$K = \frac{1}{E} = \frac{1}{\rho}\left(\frac{d\rho}{dp}\right) = -\frac{1}{\forall}\left(\frac{d\forall}{dp}\right) \qquad (2.6)$$

    - K is often expressed in terms of specific volume ∀.
- For any gaseous substance, a change in pressure is generally associated with a change in volume and a change in temperature simultaneously. A **functional relationship between the pressure, volume and temperature at any equilibrium state is known as thermodynamic equation of state for the gas.**

  For an ideal gas, the thermodynamic equation of state is given by

$$p = \rho RT \qquad (2.7)$$

-
- where *T* is the temperature in absolute thermodynamic or gas temperature scale (which are, in fact, identical), and *R* is known as the characteristic gas constant, the value of which depends upon a particular gas. However, this equation is also valid for the real gases which are thermodynamically far from their liquid phase. For air, the value of R is 287 J/kg K.

- K and E generally depend on the nature of process

1.7 Units and Dimensions:

Pascal (N/m^2) is the unit of pressure.

Pressure is usually expressed with reference to either absolute zero pressure (a complete vacuum)or local atmospheric pressure.

- The absolute pressure: It is the difference between the value of the pressure and the absolute zero pressure.

$$p_{abs} = p - 0 = p$$

- Gauge pressure: It is the diference between the value of the pressure and the local atmospheric pressure(p_{atm})

$$p_{gauge} = p - p_{atm}$$

- Vacuum Pressure: If $p<p_{atm}$ then the gauge pressure (p_{gauge}) becomes negative and is called the vacuum pressure.But one should always remember that hydrostatic pressure is always compressive in nature

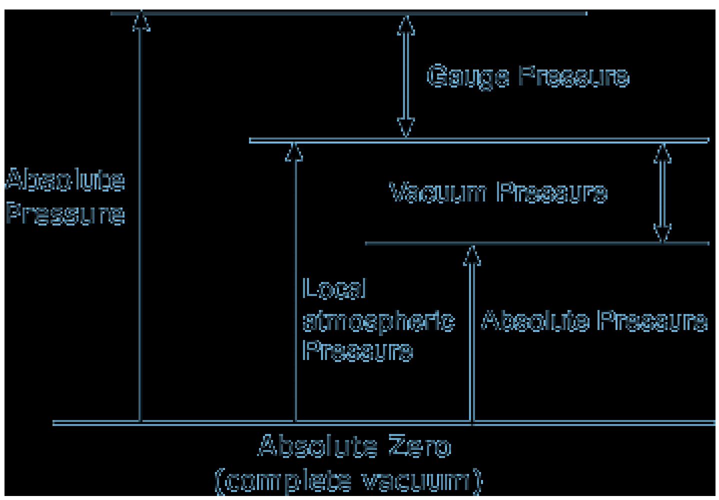


Fig 4.1 The Scale of Pressure

At sea-level, the international standard atmosphere has been chosen as P_{atm} = 101.32 kN/m^2

1.8 Normal and Shear Stresses in Fluid Flow:

Fluid Elements - Definition: Fluid element can be defined as an infinitesimal region of the fluid continuum in isolation from its surroundings.

Two types of forces exist on fluid elements

- **Body Force**: distributed over the entire mass or volume of the element. It is usually expressed per unit mass of the element or medium upon which the forces act.
  Example: Gravitational Force, Electromagnetic force fields etc.
- **Surface Force**: Forces exerted on the fluid element by its surroundings through direct contact at the surface.
  Surface force has two components:
  - Normal Force: along the normal to the area
  - Shear Force: along the plane of the area.

The ratios of these forces and the elemental area in the limit of the area tending to zero are called the normal and shear stresses respectively.

The shear force is zero for any fluid element at rest and hence the only surface force on a fluid element is the normal component.

Normal Stress in a Stationary Fluid

Consider a stationary fluid element of tetrahedral shape with three of its faces coinciding with the coordinate planes x, y and z.

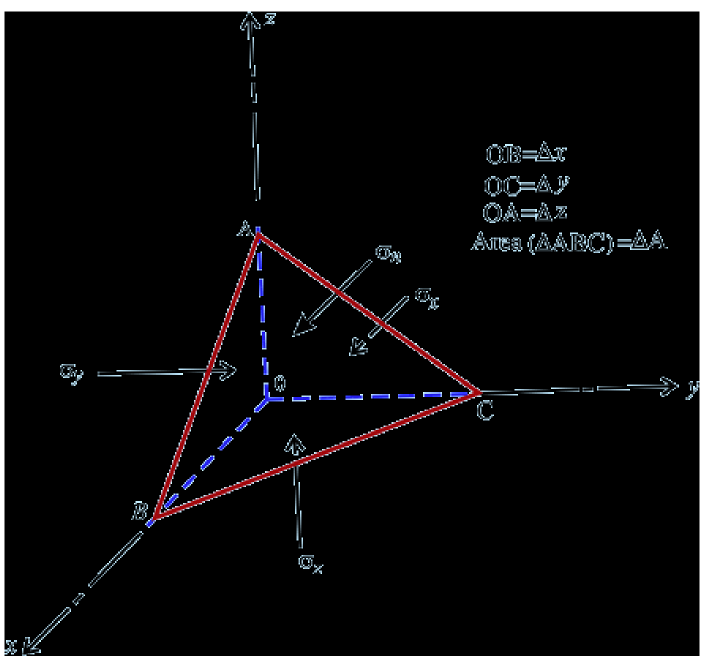


Fig 3.1 State of Stress in a Fluid Element at Rest

Since a fluid element at rest can develop neither shear stress nor tensile stress, the normal stresses acting on different faces are compressive in nature.

Suppose, ΣF_x, ΣF_y and ΣF_z are the net forces acting on the fluid element in positive x,y and z directions respectively. The direction cosines of the normal to the inclined plane of an area ΔA are cos α, cos β and cos g .Considering gravity as the only source of external body force, acting in the -ve z direction, the equations of static equlibrium for the tetrahedronal fluid element can be written as

$$\sum F_x = \sigma_x\left(\frac{\Delta y \Delta z}{2}\right) - \sigma_N \Delta A \cos\alpha = 0 \tag{3.1}$$

$$\sum F_y = \sigma_y\left(\frac{\Delta x \Delta z}{2}\right) - \sigma_N \Delta A \cos\beta = 0 \tag{3.2}$$

$$\sum F_z = \sigma_z\left(\frac{\Delta y \Delta x}{2}\right) - \sigma_N \Delta A \cos\gamma - \frac{\rho g}{6}(\Delta x \Delta y \Delta z) = 0 \tag{3.3}$$

where $\left(\frac{\Delta x \, \Delta y \, \Delta z}{6}\right)$= Volume of tetrahedral fluid element

Chapter 2
Regimes of Fluid Flow

2.1 Continuum and Free Molecular flow:

Concept of Continuum

- The concept of continuum is a kind of idealization of the continuous description of matter where the properties of the matter are considered as continuous functions of space variables. Although any matter is composed of several molecules, the concept of continuum assumes a continuous distribution of mass within the matter or system with no empty space, instead of the actual conglomeration of separate molecules.

- Describing a fluid flow quantitatively makes it necessary to assume that flow variables (pressure , velocity etc.) and fluid properties vary continuously from one point to another. Mathematical description of flow on this basis have proved to be reliable and treatment of fluid medium as a continuum has firmly become established. For example density at a point is normally defined as

$$\rho = \lim_{\Delta\forall \to 0}\left(\frac{m}{\Delta\forall}\right)$$

Here $\Delta\forall$ is the volume of the fluid element and m is the mass

- If $\Delta\forall$ is very large ρ is affected by the inhomogeneities in the fluid medium. Considering another extreme if $\Delta\forall$ is very small, random movement of atoms (or molecules) would change their number at different times. In the continuum approximation point density is defined at the smallest magnitude of $\Delta\forall$, before statistical fluctuations become significant. This is called continuum limit and is denoted by $\Delta\forall_c$.

$$\rho = \lim_{\Delta\forall \to \Delta\forall_c}\left(\frac{m}{\Delta\forall}\right)$$

- One of the factors considered important in determining the validity of continuum model is molecular density. It is the distance between the molecules which is characterised by mean free path (λ). It is calculated by finding statistical average distance the molecules travel between two successive collisions. If the mean free path is very small as compared with some characteristic length in the flow domain (i.e., the molecular density is very high) then the gas can be treated as a continuous medium. If the mean free path is large in comparison to some characteristic length, the gas cannot be considered continuous and it should be analysed by the molecular theory.

- A dimensionless parameter known as Knudsen number, $K_n = \lambda / L$, where λ is the mean free path *and* L is the characteristic length. It describes the degree of departure from continuum.

  Usually when $K_n > 0.01$, the concept of continuum does not hold good.

  Beyond this critical range of Knudsen number, the flows are known as

  slip flow ($0.01 < K_n < 0.1$),

  transition flow ($0.1 < K_n < 10$) and

  free-molecule flow ($K_n > 10$).

However, for the flow regimes considered in this course **, K_n is always less than 0.01 and it is usual to say that the fluid is a continuum.**

Other factor which checks the validity of continuum is the elapsed time between collisions. The time should be small enough so that the random statistical description of molecular activity holds good.

In continuum approach, fluid properties such as density, viscosity, thermal conductivity, temperature, etc. can be expressed as continuous functions of space and time.

2.2 Inviscid and Viscous Flows:

Refer to Section 1.3

2.3 Incompressible and Compressible Flows:

Distinction between an Incompressible and a Compressible Flow

- In order to know, if it is necessary to take into account the compressibility of gases in fluid flow problems, we need to consider whether the change in pressure brought about by the fluid motion causes large change in volume or density.

  Using Bernoulli's equation

  $p + (1/2)\rho V^2 =$ constant (*V* being the velocity of flow), change in pressure, Δp, in a flow

field, is of the order of $(1/2)\rho V^2$ (dynamic head).
Invoking this relationship into

$$E = \lim_{\delta\rho \to 0} \left(\frac{\Delta p}{\Delta \rho / \rho} \right) = \rho \frac{dp}{d\rho}$$

- we get ,

$$\frac{\Delta \rho}{\rho} \approx \frac{1}{2} \frac{\rho V^2}{E} \qquad (2.12)$$

- **So if $\Delta\rho/\rho$ is very small, the flow of gases can be treated as incompressible with a good degree of approximation.**

- According to Laplace's equation, the velocity of sound is given by

$$a = \sqrt{\frac{E}{\rho}}$$

- Hence

$$\frac{\Delta \rho}{\rho} \approx \frac{1}{2} \frac{V^2}{a^2} \approx \frac{1}{2} Ma^2$$

- where, Ma is the ratio of the velocity of flow to the acoustic velocity in the flowing medium at the condition and is known as **Mach number**. So we can conclude that the compressibility of gas in a flow can be neglected if $\Delta\rho/\rho$ is considerably smaller than unity, i.e. $(1/2)Ma^2 << 1$.

- In other words, if the flow velocity is small as compared to the local acoustic velocity, compressibility of gases can be neglected. **Considering a maximum relative change in density of 5 per cent as the criterion of an incompressible flow, the upper limit of Mach number becomes approximately 0.33**. In the case of air at standard pressure and temperature, the acoustic velocity is about 335.28 m/s. Hence a Mach number of 0.33 corresponds to a velocity of about 110 m/s. Therefore flow of air up to a velocity of 110 m/s under standard condition can be considered as incompressible flow.

2.4 Newtonian and Non- Newtonian Flow:

Ideal Fluid

- Consider a hypothetical fluid having a zero viscosity ($\mu = 0$). Such a fluid is called an *ideal fluid* and the resulting motion is called as **ideal** or **inviscid flow**. **In an ideal flow, there is no existence of shear force because of vanishing viscosity.**

$$\tau = \mu \frac{du}{dy} = 0 \qquad \text{since } \mu = 0$$

- All the **fluids in reality have viscosity** ($\mu > 0$) and hence they are termed as real fluid and their motion is known as viscous flow.

- Under certain situations of very high velocity flow of viscous fluids, an accurate analysis of flow field away from a solid surface can be made from the ideal flow theory.

Non-Newtonian Fluids

- There are certain fluids where the linear relationship between the shear stress and the deformation rate (velocity gradient in parallel flow) as expressed by the $\tau = \mu \frac{du}{dy}$ is not valid. For these fluids the viscosity varies with rate of deformation.

- Due to the deviation from Newton's law of viscosity they are commonly termed as **non-Newtonian fluids**. Figure 2.1 shows the class of fluid for which this relationship is nonlinear.

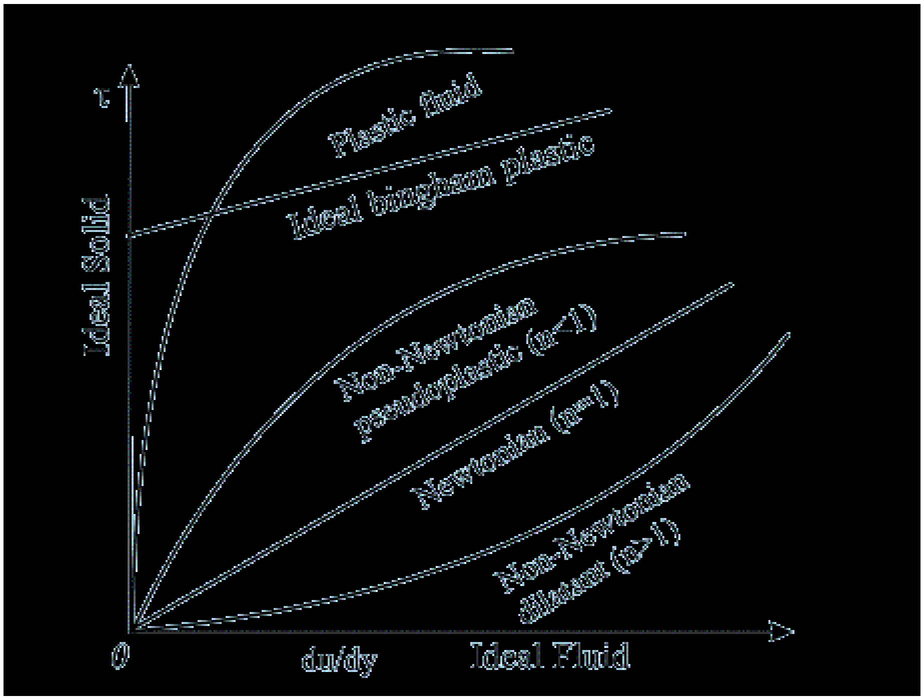


Figure 2.1 Shear stress and deformation rate relationship of different fluids

- The abscissa in Fig. 2.1 represents the behaviour of ideal fluids since for the ideal fluids the resistance to shearing deformation rate is always zero, and hence they exhibit zero shear stress under any condition of flow.
- The ordinate represents the ideal solid for there is no deformation rate under any loading condition.
- The Newtonian fluids behave according to the law that shear stress is linearly proportional to velocity gradient or rate of shear strain $\tau = \mu du/dy$. Thus for these fluids, the plot of shear stress against velocity gradient is a straight line through the origin. The slope of the line determines the viscosity.
- The non-Newtonian fluids are further classified as pseudo-plastic, dilatant and Bingham plastic.

2.5 Aerodynamic Force and Moments:

A fluid motion, under all such forces is characterised by

1. Hydrodynamic parameters like pressure, velocity and acceleration due to gravity,
2. Rheological and other physical properties of the fluid involved, and
3. Geometrical dimensions of the system.

It is important to express the magnitudes of different forces in terms of these parameters, to know the extent of their influences on the different forces acting on a flluid element in the course of its flow.

Inertia Force $\vec{F}_I$

- The inertia force acting on a fluid element is equal in magnitude to the mass of the element multiplied by its acceleration.
- The mass of a fluid element is proportional to ρl^3 where, ρ is the density of fluid and l is the characteristic geometrical dimension of the system.
- The acceleration of a fluid element in any direction is the rate at which its velocity in that direction changes with time and is therefore proportional in magnitude to some characteristic velocity V divided by some specified interval of time t. The time interval t is proportional to the characteristic length l divided by the characteristic velocity V, so that the acceleration becomes proportional to V^2/l.

The magnitude of inertia force is thus proportional to

$$\boxed{\rho l^3 V^2 / l = \rho l^2 V^2}$$

This can be written as,

$$\left|\vec{F}_I\right| \propto \rho l^2 V^2 \qquad (18.1a)$$

Viscous Force $\vec{F}_v$

The viscous force arises from shear stress in a flow of fluid.

Therefore, we can write

Magnitude of viscous force $\vec{F}_v$ = shear stress X surface area over which the shear stress acts

Again, shear stress = μ (viscosity) X rate of shear strain

where, rate of shear strain $\propto$ velocity gradient $\propto \frac{V}{l}$ and surface area $\propto l^2$

Hence

$$|\vec{F}_v| \propto \mu \frac{V}{l} l^2$$

$$\propto \mu V l \qquad (18.1b)$$

Pressure Force $\vec{F}_p$

The pressure force arises due to the difference of pressure in a flow field.

Hence it can be written as

$$|\vec{F}_p| \propto \Delta p\, l^2 \qquad (18.1c)$$

(where, Dp is some characteristic pressure difference in the flow.)

Gravity Force $\vec{F}_g$

The gravity force on a fluid element is its weight. Hence,

$$|\vec{F}_g| \propto \rho l^3 g \qquad (18.1d)$$

(where g is the acceleration due to gravity or weight per unit mass)

Capillary or Surface Tension Force $\vec{F}_s$

The capillary force arises due to the existence of an interface between two fluids.

- The surface tension force acts tangential to a surface .
- It is equal to the coefficient of surface tension σ multiplied by the length of a linear element on the surface perpendicular to which the force acts.

Therefore,

$$|\vec{F}_s| \propto \sigma l \qquad (18.1e)$$

Compressibility or Elastic Force $\vec{F}_e$

Elastic force arises due to the compressibility of the fluid in course of its flow.

- For a given compression (a decrease in volume), the increase in pressure is proportional to the bulk modulus of elasticity E
- This gives rise to a force known as the elastic force.

Hence, for a given compression $\Delta p \propto E$

$$|\vec{F}_e| \propto E^2 \qquad (18.1f)$$

The flow of a fluid in practice does not involve all the forces simultaneously.

Therefore, the pertinent dimensionless parameters for dynamic similarity are derived from the ratios of significant forces causing the flow.

2.6 Dimensional Analysis:

Principles of Physical Similarity - An Introduction

Laboratory tests are usually carried out under altered conditions of the operating variables from the actual ones in practice. These variables, in case of experiments relating to fluid flow, are pressure, velocity, geometry of the working systems and the physical properties of the working fluid.

The pertinent questions arising out of this situation are:

1. How to apply the test results from laboratory experiments to the actual problems?
2. Is it possible, to reduce the large number of experiments to a lesser one in achieving the same objective?

Answer of the above two questions lies in the principle of physical similarity. This principle is useful for the following cases:

1. To apply the results taken from tests under one set of conditions to another set of conditions

and

2. To predict the influences of a large number of independent operating variables on the performance of a system from an experiment with a limited number of operating variables.

Concept and Types of Physical Similarity

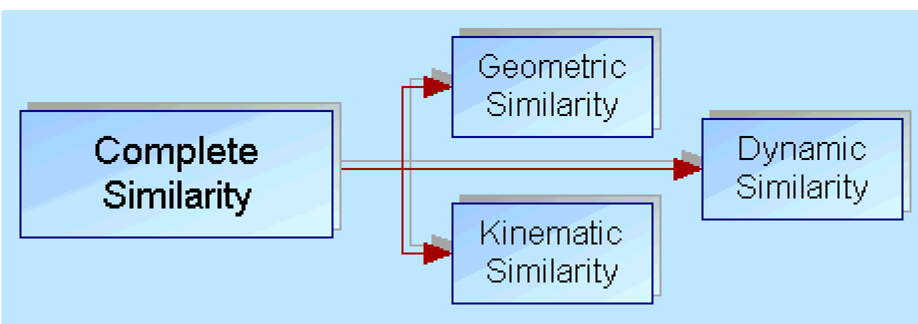
The primary and fundamental requirement for the **physical similarity** between two problems is that the **physics of the problems must be the same**.

For an example, two flows: one governed by viscous and pressure forces while the other by gravity force cannot be made physically similar. Therefore, the laws of similarity have to be sought between problems described by the same physics.

Definition of physical similarity as a general proposition

Two systems, described by the same physics, operating under different sets of conditions are said to be physically similar in respect of certain specified physical quantities; when the ratio of corresponding magnitudes of these quantities between the two systems is the same everywhere.

In the field of mechanics, there are three types of similarities which constitute the complete similarity between problems of same kind.



Geometric Similarity : If the specified physical quantities are geometrical dimensions, the similarity is called Geometric Similarty,

Kinematic Similarity : If the quantities are related to motions, the similarity is called Kinematic Similarity

Dynamic Similarity: If the quantities refer to forces, then the similarity is termed as Dynamic Similarity.

Geometric Similarity

- Geometric Similarity implies the similarity of shape such that, the **ratio of any length in one system to the corresponding length in other system is the same everywhere**.
- This ratio is usually known as **scale factor**.

Therefore, geometrically similar objects are similar in their shapes, i.e., proportionate in their physical dimensions, but differ in size.

In investigations of physical similarity,

· the full size or **actual scale systems** are known as **prototypes**

· the **laboratory scale systems** are referred to as **models**

· use of the same fluid with both the prototype and the model is not necessary

· model need not be necessarily smaller than the prototype. The flow of fluid through an injection nozzle or a carburettor , for example, would be more easily studied by using a model much larger than the prototype.

· the model and prototype may be of identical size, although the two may then differ in regard to other factors such as velocity, and properties of the fluid.

If l_1 and l_2 are the two characteristic physical dimensions of any object, then the requirement of geometrical similarity is

$$\frac{l_{1m}}{l_{1p}} = \frac{l_{2m}}{l_{2p}} = l_r$$

(**model ratio**)

(The second suffices *m* and *p* refer to model and prototype respectively) where l_r is the scale factor or sometimes known as the model ratio. Figure 5.1 shows three pairs of geometrically similar objects, namely, a right circular cylinder, a parallelopiped, and a triangular prism.

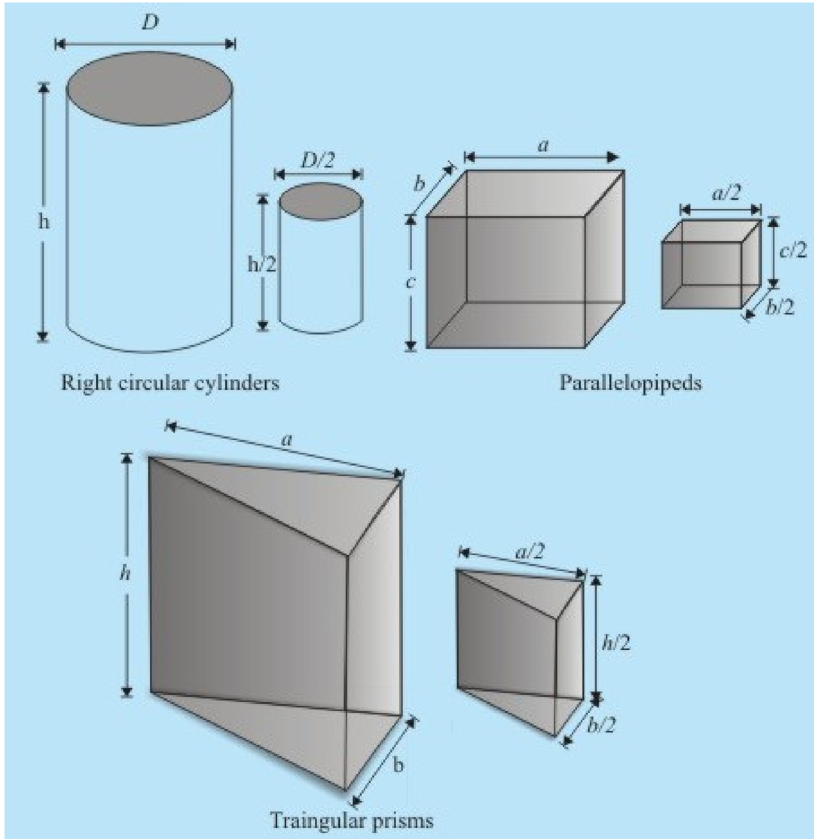


Fig 17.1 Geometrically Similar Objects (In all the above cases model ratio is ½)

Geometric similarity is perhaps the most obvious requirement in a model system designed to correspond to a given prototype system.

A perfect geometric similarity is not always easy to attain. **Problems in achieving perfect geometric similarity** are:

- For a small model, the surface roughness might not be reduced according to the scale factor (unless the model surfaces can be made very much smoother than those of the prototype). If for any reason the scale factor is not the same throughout, a distorted model results.

- Sometimes it may so happen that to have a perfect geometric similarity within the available laboratory space, physics of the problem changes. For example, in case of large prototypes, such as rivers, the size of the model is limited by the available floor space of the laboratory; but if a very low scale factor is used in reducing both the horizontal and vertical lengths, this may result in a stream so shallow that surface tension has a considerable effect and, moreover, the flow may be laminar instead of turbulent. In this situation, a distorted model may be unavoidable (a lower scale factor"for horizontal lengths while a relatively higher scale factor for vertical lengths. The extent to which perfect geometric similarity should be sought therefore depends on the problem being investigated, and the accuracy required from the solution.

Kinematic Similarity

Kinematic similarity refers to **similarity of motion.**

Since motions are described by distance and time, it implies **similarity of lengths (i.e., geometrical similarity**) and, in addition**, similarity of time intervals**.

If the corresponding lengths in the two systems are in a fixed ratio, the velocities of corresponding particles must be in a fixed ratio of magnitude of corresponding time intervals.

If the ratio of corresponding lengths, known as the **scale factor, is l_r** and **the ratio of corresponding time intervals is t_r**, then the magnitudes of corresponding **velocities are in the ratio l_r/t_r** and the magnitudes of corresponding **accelerations are in the ratio l_r/t^2_r**.

A well-known **example** of kinematic similarity is found in a planetarium. Here the galaxies of stars and planets in space are reproduced in accordance with a certain length scale and in simulating the motions of the planets, a fixed ratio of time intervals (and hence velocities and accelerations) is used.

When fluid motions are kinematically similar, the **patterns formed by streamlines are geometrically similar** at corresponding times.

Since the impermeable boundaries also represent streamlines, **kinematically similar flows are possible only past geometrically similar boundaries.**

Therefore, **geometric similarity is a necessary condition for the kinematic similarity** to be achieved, but not the sufficient one.

For example, geometrically similar boundaries may ensure geometrically similar streamlines in the near vicinity of the boundary but not at a distance from the boundary.

Dynamic Similarity

Dynamic similarity is the **similarity of forces** .

In dynamically similar systems, the **magnitudes of forces** at correspondingly similar points in each system are **in a fixed ratio.**

In a system involving flow of fluid, different forces due to different causes may act on a fluid element. These forces are as follows:

Viscous Force (due to viscosity)	$\vec{F}_v$
Pressure Force (due to different in pressure)	$\vec{F}_p$
Gravity Force (due to gravitational attraction)	$\vec{F}_g$
Capillary Force (due to surface tension)	$\vec{F}_c$
Compressibility Force (due to elasticity)	$\vec{F}_e$

According to Newton 's law, the resultant F_R of all these forces, will cause the acceleration of a fluid element. Hence

$$\vec{F}_R = \vec{F}_v + \vec{F}_p + \vec{F}_g + \vec{F}_c + \vec{F}_e \qquad (17.1)$$

Moreover, the **inertia force** $\vec{F}_i$ **is defined as equal and opposite to the resultant accelerating force** $\vec{F}_R$

$$\vec{F}_i = -\vec{F}_R$$

Therefore Eq. 17.1 can be expressed as

$$\vec{F}_v + \vec{F}_p + \vec{F}_g + \vec{F}_c + \vec{F}_e + \vec{F}_i = 0$$

For dynamic similarity, the magnitude ratios of these forces have to be same for both the prototype and the model. **The inertia force** $\vec{F}_i$ **is usually taken as the common one to describe the ratios** as (or putting in other form we equate the the non dimensionalised forces in the two systems)

$$\frac{|\vec{F}_v|}{|\vec{F}_i|}, \frac{|\vec{F}_p|}{|\vec{F}_i|}, \frac{|\vec{F}_g|}{|\vec{F}_i|}, \frac{|\vec{F}_c|}{|\vec{F}_i|}, \frac{|\vec{F}_e|}{|\vec{F}_i|}$$

2.7 Non Dimensional Parameters:

The criterion of dynamic similarity for the **flows controlled by viscous, pressure and inertia forces** are derived from the ratios of the representative magnitudes of these forces with the help of Eq. (18.1a) to (18.1c) as follows:

$$\frac{\text{Viscous Force}}{\text{Inertia Force}} = \frac{|\vec{F}_v|}{|\vec{F}_i|} \propto \frac{\mu V l}{\rho V^2 l^2} = \frac{\mu}{\rho V l} \qquad (18.2a)$$

$$\frac{\text{Pressure Force}}{\text{Inertia Force}} = \frac{|\vec{F}_p|}{|\vec{F}_i|} \propto \frac{\Delta p l^2}{\rho V^2 l^2} = \frac{\Delta p}{\rho V^2} \qquad (18.2b)$$

The term $\rho V l/\mu$ **is known as Reynolds number, Re** after the name of the scientist who first developed it and is thus proportional to the magnitude ratio of inertia force to viscous force .(Reynolds number plays a vital role in the analysis of fluid flow)

The term $\Delta p/\rho V^2$ **is known as Euler number, Eu** after the name of the scientist who first derived it. The dimensionless terms Re and Eu represent the critieria of dynamic similarity for the flows which are affected only by viscous, pressure and inertia forces. Such instances, for example, are

1. the full flow of fluid in a completely closed conduit,
2. flow of air past a low-speed aircraft and
3. the flow of water past a submarine deeply submerged to produce no waves on the surface.

Hence, for a complete dynamic similarity to exist between the prototype and the model for this class of flows, the Reynolds number, Re and Euler number, Eu have to be same for the two (prototype and model). Thus

$$\frac{\rho_p l_p V_p}{\mu_p} = \frac{\rho_m l_m V_m}{\mu_m} \qquad (18.2c)$$

$$\frac{\Delta p_p}{\rho_p V_p^2} = \frac{\Delta p_m}{\rho_m V_m^2} \qquad (18.2d)$$

where, the suffix p and suffix m refer to the parameters for prototype and model respectively.

In practice, the pressure drop is the dependent variable, and hence it is compared for the two systems with the help of Eq. (18.2d), while the equality of Reynolds number (Eq. (18.2c)) along with the equalities of other parameters in relation to kinematic and geometric similarities are maintained.

- The characteristic geometrical dimension l **and** the reference velocity **V** in the expression of the Reynolds number **may be any geometrical dimension and any velocity which are significant in determining the pattern of flow.**

- For internal flows through a closed duct, the hydraulic diameter of the duct D_h and the average flow velocity at a section are invariably used for l and V respectively.
- The hydraulic diameter D_h is defined as $D_h = 4A/P$ where A and P are the cross-sectional area and wetted perimeter respectively.

A flow of the type in which **significant forces** are **gravity force, pressure force and inertia force,** is found **when a free surface is present.**

Examples can be

1. the flow of a liquid in an open channel.
2. the wave motion caused by the passage of a ship through water.
3. the flows over weirs and spillways.

The condition for dynamic similarity of such flows requires

- the equality of the Euler number Eu (the magnitude ratio of pressure to inertia force), and
- the equality of the magnitude ratio of gravity to inertia force at corresponding points in the systems being compared. Thus ,

$$\frac{\text{Gravity Force}}{\text{Inertia Force}} = \left|\frac{\vec{F}_g}{\vec{F}_I}\right| \propto \frac{\rho l^3 g}{\rho V^2 l^2} = \frac{lg}{V^2} \qquad (18.2e)$$

- In practice, it is often convenient to use the square root of this ratio so to deal with the first power of the velocity.
- From a physical point of view, equality of $(lg)^{1/2}/V$ implies equality of lg/V^2 as regard to the concept of dynamic similarity.

The **reciprocal of the term** $(lg)^{1/2}/V$ **is known as Froude number (** after William Froude who first suggested the use of this number in the study of naval architecture.)

Hence **Froude number,** $Fr = V/(lg)^{1/2}$.

Therefore, **the primary requirement for dynamic similarity between the prototype and the model involving flow of fluid with gravity as the significant force, is the equality of Froude number, Fr,** i.e.,

$$\frac{(l_p g_p)^{1/2}}{V_p} = \frac{(l_m g_m)^{1/2}}{V_m} \qquad (18.2f)$$

Surface tension forces are important in certain classes of practical problems such as ,

1. flows in which capillary waves appear
2. flows of small jets and thin sheets of liquid injected by a nozzle in air
3. flow of a thin sheet of liquid over a solid surface.

Here the significant parameter for dynamic similarity is the magnitude ratio of the surface tension force to the inertia force.

This can be written as $$\frac{|\vec{F}_s|}{|\vec{F}_I|} \propto \frac{\sigma l}{\rho V^2 l^2} = \frac{\sigma}{\rho V^2 l} \qquad (18.2g)$$

The term $\sigma/\rho V^2 l$ **is usually known as Weber number, Wb** (after the German naval architect Moritz Weber who first suggested the use of this term as a relevant parameter.)

Thus for dynamically similar flows $(Wb)_m = (Wb)_p$

i.e., $$\frac{\sigma_m}{\rho_m V_m^2 L_m} = \frac{\sigma_p}{\rho_p V_p^2 L_p^2}$$

When the compressibility of fluid in the course of its flow becomes significant, the elastic force along with the pressure and inertia forces has to be considered.

Therefore, the magnitude ratio of inertia to elastic force becomes a relevant parameter for dynamic similarity under this situation.

Thus we can write,

$$\boxed{\frac{\text{Inertia force}}{\text{Elastic Force}} = \frac{|\vec{F}_I|}{|\vec{F}_e|} \propto \frac{\rho V^2 l^2}{E l^2} = \frac{\rho V^2}{E}} \qquad (18.2h)$$

The parameter $\rho V^2/E$ **is known as Cauchy number, (**after the French mathematician A.L. Cauchy)

If we consider the **flow to be isentropic,** then it can be written

$$\frac{|\vec{F}_I|}{|\vec{F}_e|} \propto \frac{\rho V^2}{E_s} \qquad (18.2i)$$

(where E_s is the isentropic bulk modulus of elasticity)

Thus for dynamically similar flows $(\text{cauchy})_m = (\text{cauchy})_p$

ie., $$\frac{\rho_m V_m^2}{(E_s)_m} = \frac{\rho_p V_p^2}{(E_s)_p}$$

- The velocity with which a sound wave propagates through a fluid medium equals to $\sqrt{E_s/\rho}$.
- Hence, the term $\rho V^2/E_s$ can be written as V^2/a^2 where **a is the acoustic velocity** in the fluid medium.

The ratio V/a is known as Mach number, Ma (after an Austrian physicist Earnst Mach)

It has been shown in Chapter 1 that the effects of compressibility become important when the Mach number exceeds 0.33.

The situation arises in the flow of air past high-speed aircraft, missiles, propellers and rotory compressors. In these cases equality of Mach number is a condition for dynamic similarity. Therefore,

$(\text{Ma})_p = (\text{Ma})_m$

i.e.

$$\boxed{V_p/a_p = V_m/a_m} \qquad (18.2j)$$

Ratios of Forces for Different Situations of Flow

Pertinent Dimensionless term as the croterion of dynamic similarity in different situations of fluid flow	Representative magnitude ration of the forces	Name	Recommended symbol
$\rho l V/\mu$	$\frac{\text{Inertia force}}{\text{Viscous force}}$	Reynolds number	Re
$\Delta p/\rho V^2$	$\frac{\text{Pressure force}}{\text{Inertia force}}$	Euler number	Eu
$V/(lg)^{1/2}$	$\frac{\text{Inertia force}}{\text{Gravity force}}$	Froude number	Fr

$\sigma / \rho V^2 l$	$\dfrac{\text{Surface Tesion force}}{\text{Inertia force}}$	Weber number	Wb
$V / \sqrt{E_v / \rho}$	$\dfrac{\text{Inertia force}}{\text{Elastic force}}$	Mach number	Ma

Chapter 3
Fluid Statics

3.1 Pascal's Law:

Pascal's Law of Hydrostatics

Pascal's Law

The normal stresses at any point in a fluid element at rest are directed towards the point from all directions and they are of the equal magnitude.

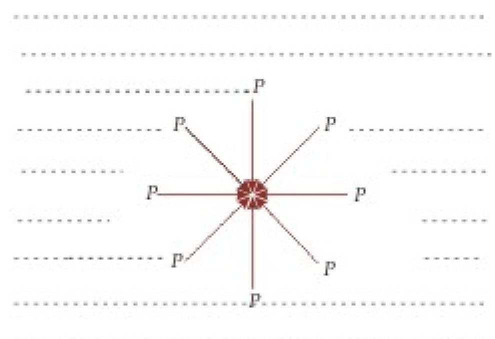


Fig 3.2 State of normal stress at a point in a fluid body at rest

Derivation:
The inclined plane area is related to the fluid elements (refer to Fig 3.1) as follows

$$\Delta A \cos\alpha = \left(\frac{\Delta y \Delta z}{2}\right) \quad (3.4)$$

$$\Delta A \cos\beta = \left(\frac{\Delta x \Delta z}{2}\right) \quad (3.5)$$

$$\Delta A \cos\gamma = \left(\frac{\Delta x \Delta y}{2}\right) \quad (3.6)$$

Substituting above values in equation 3.1- 3.3 we get

$$\sigma_x = \sigma_y = \sigma_z = \sigma_n \quad (3.7)$$

Conclusion:

The state of normal stress at any point in a fluid element at rest is same and directed towards the point from all directions. These stresses are denoted by a scalar quantity p defined as the hydrostatic or thermodynamic pressure.
Using "+" sign for the tensile stress the above equation can be written in terms of pressure as

$$\sigma_x = \sigma_y = \sigma_z = -p \tag{3.8}$$

3.2 Types of Forces on Fluid Systems:

Forces on Fluid Elements

Fluid Elements - Definition:

Fluid element can be defined as an infinitesimal region of the fluid continuum in isolation from its surroundings.

Two types of forces exist on fluid elements

- **Body Force**: distributed over the entire mass or volume of the element. It is usually expressed per unit mass of the element or medium upon which the forces act.
  Example: Gravitational Force, Electromagnetic force fields etc.
- **Surface Force**: Forces exerted on the fluid element by its surroundings through direct contact at the surface.
  Surface force has two components:
  - Normal Force: along the normal to the area
  - Shear Force: along the plane of the area.

The ratios of these forces and the elemental area in the limit of the area tending to zero are called the normal and shear stresses respectively.

The shear force is zero for any fluid element at rest and hence the only surface force on a fluid element is the normal component.

Normal Stress in a Stationary Fluid

Consider a stationary fluid element of tetrahedral shape with three of its faces coinciding with the coordinate planes x, y and z.

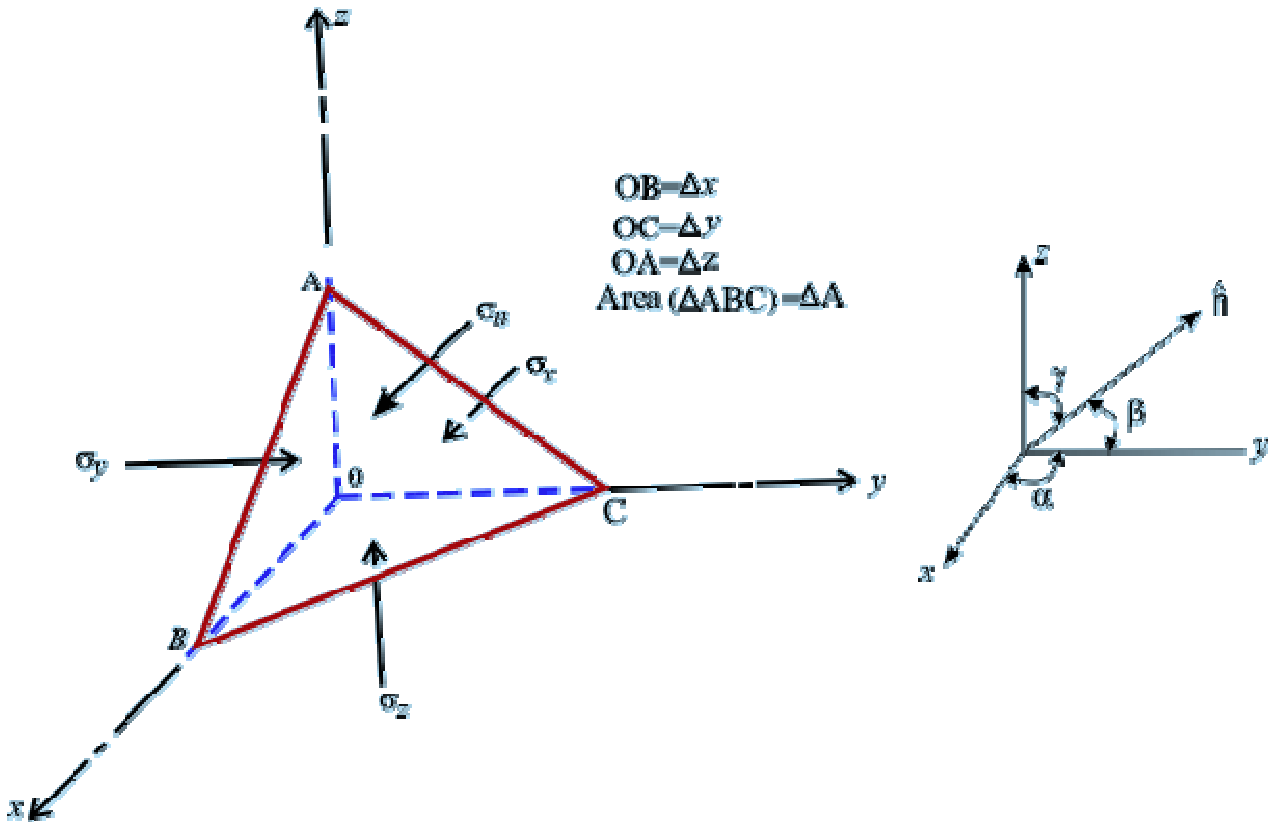


Fig 3.1 State of Stress in a Fluid Element at Rest

Since a fluid element at rest can develop neither shear stress nor tensile stress, the normal stresses acting on different faces are compressive in nature.

Suppose, ΣF_x, ΣF_y and ΣF_z are the net forces acting on the fluid element in positive x,y and z directions respectively. The direction cosines of the normal to the inclined plane of an area ΔA are cos α, cos β and cos γ .Considering gravity as the only source of external body force, acting in the -ve z direction, the equations of static equlibrium for the tetrahedronal fluid element can be written as

$$\sum F_x = \sigma_x\left(\frac{\Delta y \Delta z}{2}\right) - \sigma_n \Delta A \cos\alpha = 0 \quad (3.1)$$

$$\sum F_y = \sigma_y\left(\frac{\Delta x \Delta z}{2}\right) - \sigma_n \Delta A \cos\beta = 0 \quad (3.2)$$

$$\sum F_z = \sigma_z\left(\frac{\Delta y \Delta x}{2}\right) - \sigma_n \Delta A \cos\gamma - \frac{\rho g}{6}(\Delta x \Delta y \Delta z) = 0 \quad (3.3)$$

where $\left(\frac{\Delta x\, \Delta y\, \Delta z}{6}\right)$ = Volume of tetrahedral fluid element

3.3 Measurement of Pressure:

Pascal (N/m^2) is the unit of pressure .

Pressure is usually expressed with reference to either absolute zero pressure (a complete vacuum)or local atmospheric pressure.

- The absolute pressure: It is the difference between the value of the pressure and the absolute zero pressure.

$$p_{abs} = p - 0 = p$$

- Gauge pressure: It is the diference between the value of the pressure and the local atmospheric pressure(p_{atm})

$$p_{gauge} = p - p_{atm}$$

- Vacuum Pressure: If $p<p_{atm}$ then the gauge pressure (p_{gauge}) becomes negative and is called the vacuum pressure.But one should always remember that hydrostatic pressure is always compressive in nature

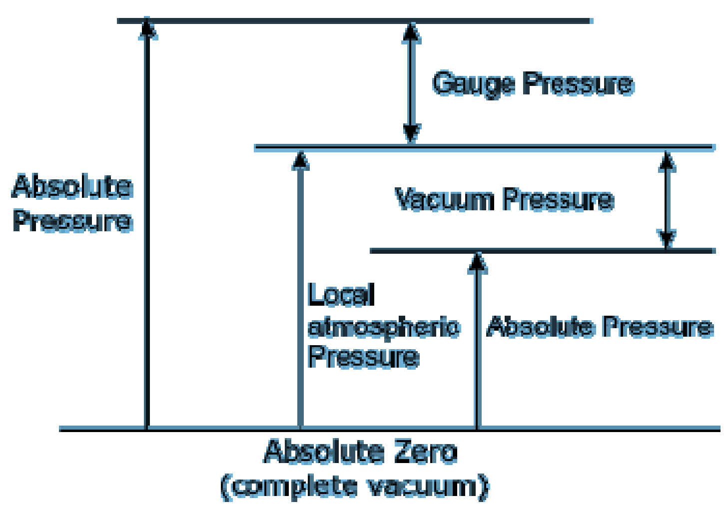


Fig 4.1 The Scale of Pressure

At sea-level, the international standard atmosphere has been chosen as P_{atm} = 101.32 kN/m^2

Piezometer Tube

The direct proportional relation between gauge pressure and the height h for a fluid of constant density enables the pressure to be simply visualized in terms of the vertical height, $h = p/\rho g$.

The height h is termed as pressure head corresponding to pressure p. For a liquid without a free surface in a closed pipe, the pressure head $p/\rho g$ at a point corresponds to the vertical height above the point to which a free surface would rise, if a small tube of sufficient length and open to atmosphere is connected to the pipe

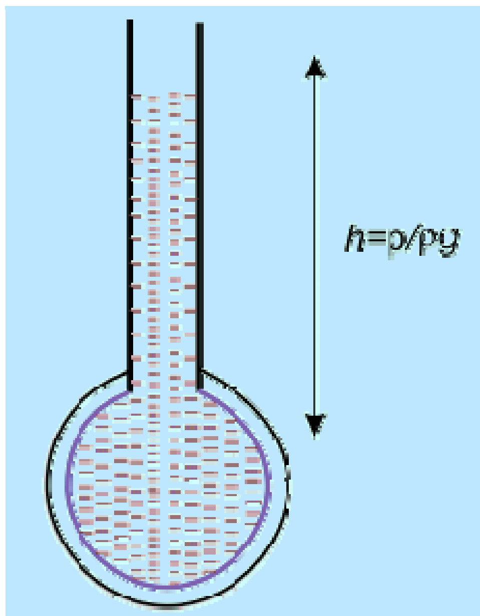


Fig 4.2 A piezometer Tube

Such a tube is called a piezometer tube, and the height h is the measure of the gauge pressure of the fluid in the pipe. If such a piezometer tube of sufficient length were closed at the top and the space above the liquid surface were a perfect vacuum, the height of the column would then correspond to the absolute pressure of the liquid at the base. This principle is used in the well known mercury barometer to determine the local atmospheric pressure.

3.4 Manometers and Gauges:

The Barometer

Barometer is used to determine the local atmospheric pressure. Mercury is employed in the barometer because its density is sufficiently high for a relative short column to be obtained. and also because it has very small vapour pressure at normal temperature. High density scales down the pressure head(h) to repesent same magnitude of pressure in a tube of smaller height.

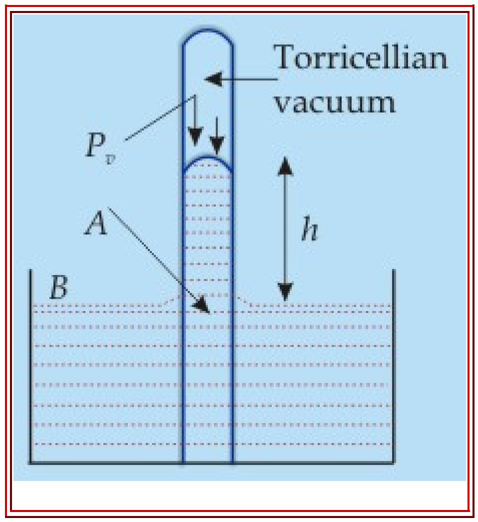


Fig 4.3 A Simple Barometer

Even if the air is completely absent, a perfect vacuum at the top of the tube is never possible. The space would be occupied by the mercury vapour and the pressure would equal to the vapour pressure of mercury at its existing temperature. This almost vacuum condition above the mercury in the barometer is known as Torricellian vacuum.

The pressure at A equal to that at B (Fig. 4.3) which is the atmospheric pressure p_{atm} since A and B lie on the same horizontal plane. Therefore, we can write

$$P_B = P_{atm} = P_v + \rho g h \qquad (4.1)$$

The vapour pressure of mercury p_v, can normally be neglected in comparison to p_{atm}.
At 20^0C,P_v is only 0.16 p_{atm}, where p_{atm} =1.0132 X10^5 Pa at sea level. Then we get from Eq. (4.1)

$$h = P_{atm}/\rho g = \frac{1.0132\times10^5\ N/m^2}{(13560 kg/m^3)(9.81 N/Kg)} = 0.732 m\ of\ Hg$$

For accuracy, small corrections are necessary to allow for the variation of r with temperature, the thermal expansion of the scale (usually made of brass). and surface tension effects. If water was used instead of mercury, the corresponding height of the column would be about 10.4 m provided that a perfect vacuum could be achieved above the water. However, the vapour pressure of water at ordinary temperature is appreciable and so the actual height at, say, 15°C would be about 180 mm less than this value. Moreover with a tube smaller in diameter than about 15 mm, surface tension effects become significant.

Manometers for measuring Gauge and Vacuum Pressure

Manometers are devices in which columns of a suitable liquid are used to measure the difference in pressure between two points or between a certain point and the atmosphere.

Manometer is needed for measuring large gauge pressures. It is basically the modified form of the piezometric tube. A common type manometer is like a transparent "U-tube" as shown in Fig. 4.4.

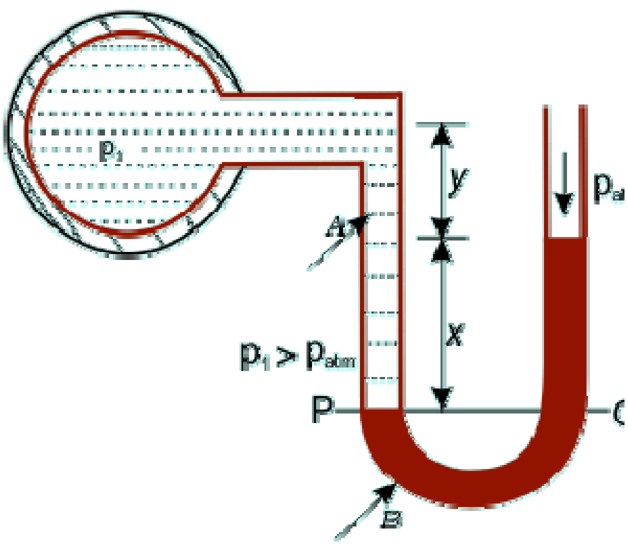


Fig 4.4 A simple manometer to measure gauge pressure

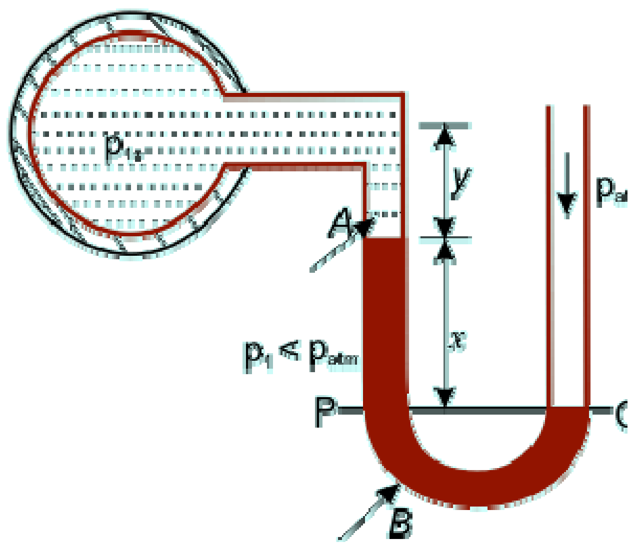


Fig 4.5 A simple manometer to measure vacuum pressure

One of the ends is connected to a pipe or a container having a fluid (A) whose pressure is to be measured while the other end is open to atmosphere. The lower part of the U-tube contains a liquid immiscible with the fluid A and is of greater density than that of A. This fluid is called the manometric fluid.

The pressures at two points *P* and *Q* (Fig. 4.4) in a horizontal plane within the continuous expanse of same fluid (the liquid B in this case) must be equal. Then equating the pressures at *P* and Q in terms of the heights of the fluids above those points, with the aid of the fundamental equation of hydrostatics (Eq 3.16), we have

$$p_1 + \rho_A g(y + x) = p_{atm} + \rho_B g x$$

Hence,

$$p_1 - p_{atm} = (\rho_B - \rho_A) g x - \rho_A g y$$

where p_1 is the absolute pressure of the fluid A in the pipe or container at its centre line, and p_{atm} is the local atmospheric pressure. When the pressure of the fluid in the container is lower than the atmospheric pressure, the liquid levels in the manometer would be adjusted as shown in Fig. 4.5. Hence it becomes,

$$p_1 + \rho_A g y + \rho_B g x = p_{atm}$$

$$p_{atm} - p_1 = (\rho_A y + \rho_B x) g \quad (4.2)$$

Manometers to measure Pressure Difference

A manometer is also frequently used to measure the pressure difference, in course of flow, across a restriction in a horizontal pipe.

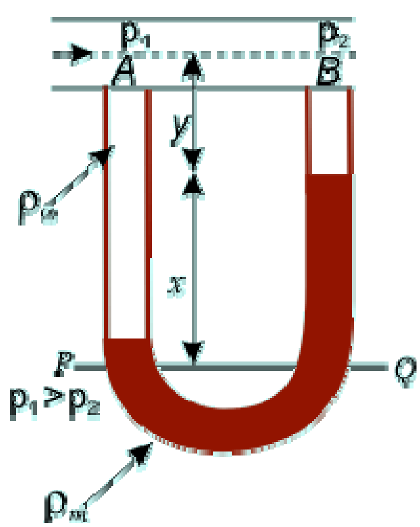


Fig 4.6 Manometer measuring pressure difference

The axis of each connecting tube at A and B should be perpendicular to the direction of flow and also for the edges of the connections to be smooth. Applying the principle of hydrostatics at P and Q we have,

$$p_1 + (y + x)\rho_w g = p_2 + y\rho_w g + \rho_m g x$$

$$p_1 - p_2 = (\rho_m - \rho_w) g x \qquad (4.3)$$

where, ρ_m is the density of manometric fluid and ρ_w is the density of the working fluid flowing through the pipe.

We can express the difference of pressure in terms of the difference of heads (height of the working fluid at equilibrium).

$$h_1 - h_2 = \frac{p_1 - p_2}{\rho_w g} = \left(\frac{\rho_m}{\rho_w} - 1\right) x \qquad (4.4)$$

Inclined Tube Manometer

- For accurate measurement of small pressure differences by an ordinary u-tube manometer, it is essential that the ratio r_m/r_w should be close to unity. This is not possible if the working fluid is a gas; also having a manometric liquid of density very close to that of the working liquid and giving at the same time a well defined meniscus at the interface is not always possible. For this purpose, an inclined tube manometer is used.

- If the transparent tube of a manometer, instead of being vertical, is set at an angle θ to the horizontal (Fig. 4.7), then a pressure difference corresponding to a vertical difference of levels x gives a movement of the meniscus s = x/sinq along the slope.

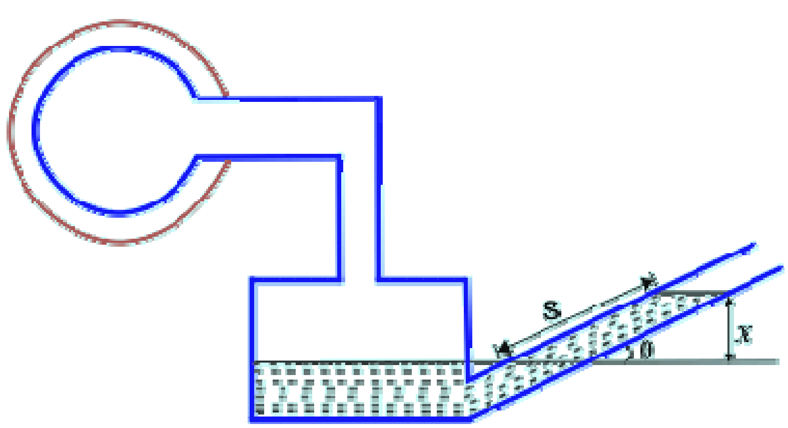


Fig 4.7 An Inclined Tube Manometer

- If θ is small, a considerable mangnification of the movement of the meniscus may be achieved.
- Angles less than 5^0 are not usually satisfactory, because it becomes difficult to determine the exact position of the meniscus.
- One limb is usually made very much greater in cross-section than the other. When a pressure difference is applied across the manometer, the movement of the liquid surface in the wider limb is practically negligible compared to that occurring in the narrower limb. If the level of the surface in the wider limb is assumed constant, the displacement of the meniscus in the narrower limb needs only to be measured, and therefore only this limb is required to be transparent.

Inverted Tube Manometer

For the measurement of small pressure differences in liquids, an inverted U-tube manometer is used.

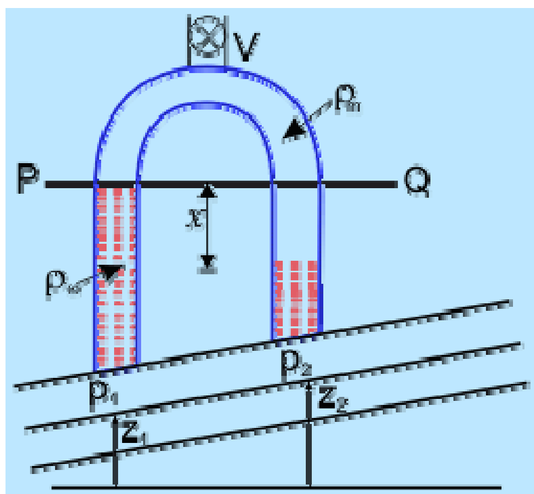


Fig 4.8 An Inverted Tube Manometer

Here $\rho_m < \rho_w$ and the line PQ is taken at the level of the higher meniscus to equate the pressures at P and Q from the principle of hydrostatics. It may be written that

$$p_1^* - p_2^* = (\rho_w - \rho_m)gx$$

where p^* represents the **piezometric pressure**, $p + \rho g z$ (z being the vertical height of the point concerned from any reference datum). In case of a horizontal pipe ($z_1 = z_2$) the difference in piezometric pressure becomes equal to the difference in the static pressure. If $(\rho_w - \rho_m)$ is sufficiently small, a large value of x may be obtained for a small value of $p_1^* - p_2^*$. Air is used as the manometric fluid. Therefore, ρ_m is negligible compared with ρ_w and hence,

$$p_1^* - p_2^* \approx \rho_w g x \tag{4.5}$$

Air may be pumped through a valve V at the top of the manometer until the liquid menisci are at a suitable level.

Micromanometer

When an additional gauge liquid is used in a U-tube manometer, a large difference in meniscus levels may be obtained for a very small pressure difference.

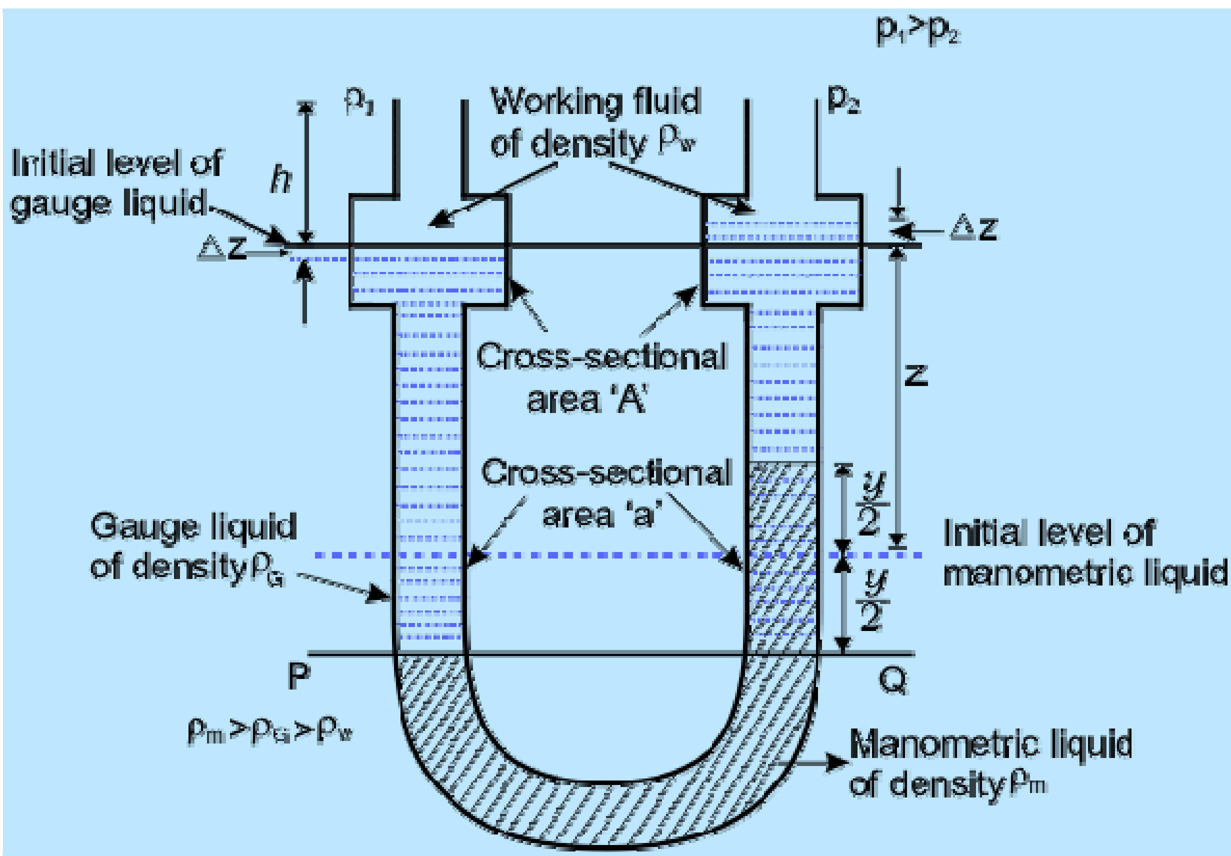


Fig 4.9 A Micromanometer

The equation of hydrostatic equilibrium at PQ can be written as

$$p_1 + \rho_w g(h + \Delta z) + \rho_G g\left(z - \Delta z + \frac{y}{2}\right) = p_2 + \rho_w g(h - \Delta z) + \rho_G g\left(z + \Delta z - \frac{y}{2}\right) + \rho_m g y$$

where ρ_w, ρ_G and ρ_m are the densities of working fluid, gauge liquid and manometric liquid respectively.
From continuity of gauge liquid,

$$A\Delta z = a\frac{y}{2} \qquad (4.6)$$

$$p_1 - p_2 = gy\left[\rho_m - \rho_G\left(1 - \frac{a}{A}\right) - \rho_w \frac{a}{A}\right] \qquad (4.7)$$

If a is very small compared to A

$$p_1 - p_2 \approx (\rho_m - \rho_G)gy \qquad (4.8)$$

With a suitable choice for the manometric and gauge liquids so that their densities are close $(\rho_m \approx \rho_G)$ a reasonable value of y may be achieved for a small pressure difference.

3.5 Hydraulic Devices: Under Development

3.6 Forces on Partially and Fully Submerged Bodies (Curved Surfces):

Hydrostatic Thrusts on Submerged Plane Surface

Due to the existence of hydrostatic pressure in a fluid mass, a normal force is exerted on any part of a solid surface which is in contact with a fluid. The individual forces distributed over an area give rise to a resultant force.

Plane Surfaces

Consider a plane surface of arbitrary shape wholly submerged in a liquid so that the plane of the surface makes an angle θ with the free surface of the liquid. We will assume the case where the surface shown in the figure below is subjected to hydrostatic pressure on one side and atmospheric pressure on the other side.

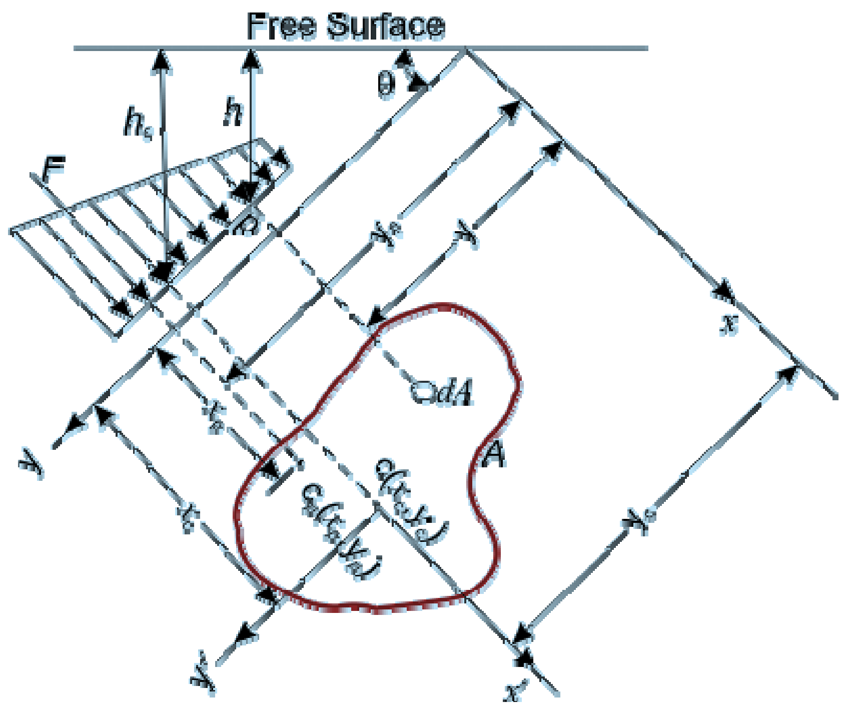


Fig 5.1 Hydrostatic Thrust on Submerged Inclined Plane Surface

Let p denotes the gauge pressure on an elemental area dA. The resultant force F on the area A is therefore

$$F = \iint_A p\,dA \tag{5.1}$$

According to Eq (3.16a) Eq (5.1) reduces to

$$F = \rho g \iint h\,dA = \rho g \sin\theta \iint y\,dA \tag{5.2}$$

Where **h is the vertical depth of the elemental area dA from the free surface** and the distance y is measured from the x-axis, the line of intersection between the extension of the inclined plane and the free surface (Fig. 5.1). The ordinate of the centre of area of the plane surface A is defined as

$$y_c = \frac{1}{A}\iint y\,dA \tag{5.3}$$

Hence from Eqs (5.2) and (5.3), we get

$$F = \rho g\, y_c \sin\theta\, A = \rho g\, h_c\, A \tag{5.4}$$

where $h_c (= y_c \sin\theta)$ is the vertical depth (from free surface) of centre c of area .

Equation (5.4) implies that the hydrostatic thrust on an inclined plane is equal to the pressure at its centroid times the total area of the surface, i.e., the force that would have been experienced by the surface if placed horizontally at a depth h_c from the free surface (Fig. 5.2).

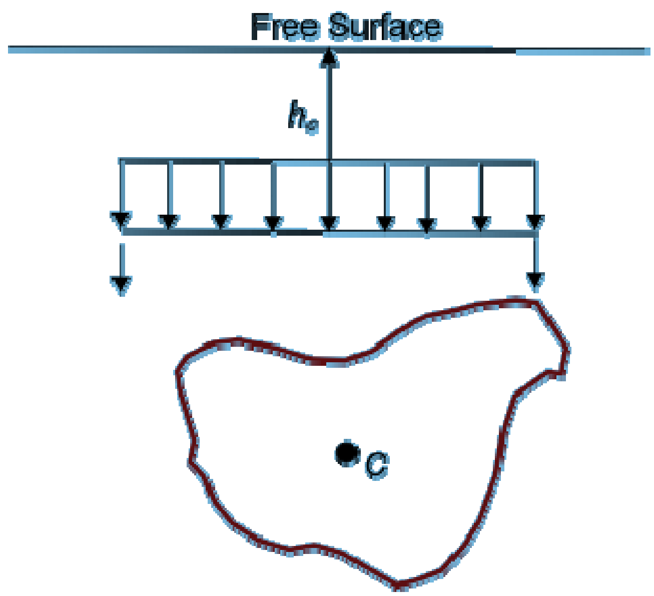


Fig 5.2 Hydrostatic Thrust on Submerged Horizontal Plane Surface

The point of action of the resultant force on the plane surface is called the centre of pressure c_p . Let x_p and y_p be the distances of the centre of pressure from the y and x axes respectively. Equating the moment of the resultant force about the x axis to the summation of the moments of the component forces, we have

$$y_p F = \int y\,dF = \rho g \sin\theta \iint y^2 dA \tag{5.5}$$

Solving for y_p from Eq. (5.5) and replacing F from Eq. (5.2), we can write

$$y_p = \frac{\iint_A y^2 dA}{\iint_A y\,dA} \tag{5.6}$$

In the same manner, the x coordinate of the centre of pressure can be obtained by taking moment about the y-axis. Therefore,

$$x_p F = \int x dF = \rho g \sin\theta \iint xy dA$$

From which,

$$x_p = \frac{\iint_A xy dA}{\iint_A y dA} \quad (5.7)$$

The two double integrals in the numerators of Eqs (5.6) and (5.7) are the moment of inertia about the x-axis $I_{xx}$and the product of inertia I_{xy} about x and y axis of the plane area respectively. By applying the theorem of parallel axis

$$I_{xx} = \iint y^2 dA = I_{x'x'} + A y_c^2 \quad (5.8)$$

$$I_{xy} = \iint xy dA = I_{x'y'} + A x_c y_c \quad (5.9)$$

where, $I_{x'x'}$ and $I_{x'y'}$ are the moment of inertia and the product of inertia of the surface about the centroidal axes $(x' - y')$, x_c, and y_c are the coordinates of the center c of the area with respect to x-y axes.

With the help of Eqs (5.8), (5.9) and (5.3), Eqs (5.6) and (5.7) can be written as

$$y_p = \frac{I_{x'x'}}{A y_c} + y_c \quad (5.10a)$$

$$x_p = \frac{I_{x'y'}}{A y_c} - x_c \quad (5.10b)$$

The first term on the right hand side of the Eq. (5.10a) is always positive. Hence, the centre of pressure is always at a higher depth from the free surface than that at which the centre of area lies. This is obvious because of the typical variation of hydrostatic pressure with the depth from the free surface. When the plane area is symmetrical about the y' axis, $I_{x'y'} = 0$, and $x_p = x_c$.

Hydrostatic Thrusts on Submerged Curved Surfaces

On a curved surface, the direction of the normal changes from point to point, and hence the pressure forces on individual elemental surfaces differ in their directions. Therefore, a scalar summation of them cannot be made. Instead, the resultant thrusts in certain directions are to be determined and these forces may then be combined vectorially. An arbitrary submerged curved surface is shown in Fig. 5.3. A rectangular Cartesian coordinate system is introduced whose xy plane coincides with the free surface of the liquid and z-axis is directed downward below the x - y plane.

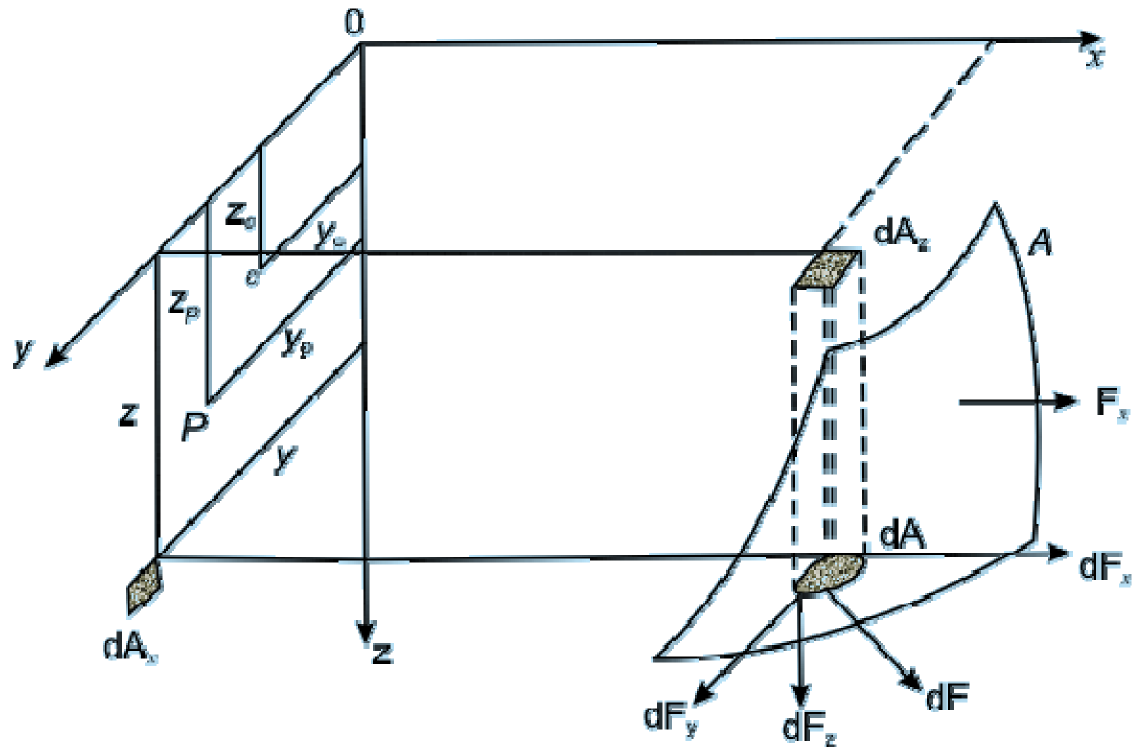


Fig 5.3 Hydrostatic thrust on a Submerged Curved Surface

Consider an elemental area dA at a depth z from the surface of the liquid. The hydrostatic force on the elemental area dA is

$$d\,F = \rho\, g\, z\, dA \qquad (5.11)$$

and the force acts in a direction normal to the area dA. The components of the force dF in x, y and z directions are

$$dF_x = l dF = l \rho g z dA \qquad (5.12a)$$

$$dF_y = m dF = m \rho g z dA \qquad (5.12b)$$

$$dF_z = n dF = n \rho g z dA \qquad (5.13c)$$

Where l, m and n are the direction cosines of the normal to dA. The components of the surface element dA projected on yz, xz and xy planes are, respectively

$$dA_x = l\,dA \tag{5.13a}$$

$$dA_y = m\,dA \tag{5.13b}$$

$$dA_z = n\,dA \tag{5.13c}$$

Substituting Eqs (5.13a-5.13c) into (5.12) we can write

$$dF_x = \rho g z\, dA_x \tag{5.14a}$$

$$dF_y = \rho g z\, dA_y \tag{5.14b}$$

$$dF_z = \rho g z\, dA_z \tag{5.14c}$$

Therefore, the components of the total hydrostatic force along the coordinate axes are

$$F_x = \iint \rho g z\, dA_x = \rho g z_c A_x \tag{5.15a}$$

$$F_y = \iint \rho g z\, dA_y = \rho g z_c A_y \tag{5.15b}$$

$$F_z = \iint \rho g z\, dA_z \tag{5.15c}$$

where z_c is the z coordinate of the centroid of area A_x and A_y (the projected areas of curved surface on yz and xz plane respectively). If z_p and y_p are taken to be the coordinates of the point of action of F_x on the projected area A_x on yz plane, , we can write

$$z_p = \frac{1}{A_x z_c}\iint z^2\, dA_x = \frac{I_{yy}}{A_x z_c} \tag{5.16a}$$

$$y_p = \frac{1}{A_x z_c}\iint yz\, dA_x = \frac{I_{yz}}{A_x z_c} \tag{5.16b}$$

where I_{yy} is the moment of inertia of area A_x about y-axis and I_{yz} is the product of inertia of A_x with respect to axes y and z. In the similar fashion, z_p' and x_p' the coordinates of the point of action of the force F_y on area A_y, can be written as

$$z'_p = \frac{1}{A_y z_c}\iint z^2 dA_y = \frac{I_{xx}}{A_y z_c} \qquad (5.17a)$$

$$x'_p = \frac{1}{A_y z_c}\iint xz\, dA_y = \frac{I_{xz}}{A_y z_c} \qquad (5.17b)$$

where I_{xx} is the moment of inertia of area A_y about x axis and I_{xz} is the product of inertia of A_y about the axes x and z.

We can conclude from Eqs (5.15), (5.16) and (5.17) that for a curved surface, the component of hydrostatic force in a horizontal direction is equal to the hydrostatic force on the projected plane surface perpendicular to that direction and acts through the centre of pressure of the projected area. From Eq. (5.15c), the vertical component of the hydrostatic force on the curved surface can be written as

$$F_z = \rho g \iint z\, dA_z = \rho g \forall \qquad (5.18)$$

where $\forall$is the volume of the body of liquid within the region extending vertically above the submerged surface to the free surfgace of the liquid. Therefore, the vertical component of hydrostatic force on a submerged curved surface is equal to the weight of the liquid volume vertically above the solid surface of the liquid and acts through the center of gravity of the liquid in that volume.

3.7 Bouyancy:

Buoyancy

- When a body is either wholly or partially immersed in a fluid, a lift is generated due to the net vertical component of hydrostatic pressure forces experienced by the body.
- This lift is called the buoyant force and the phenomenon is called buoyancy
- Consider a solid body of arbitrary shape completely submerged in a homogeneous liquid as shown in Fig. 5.4. **Hydrostatic pressure forces act on the entire surface of the body.**

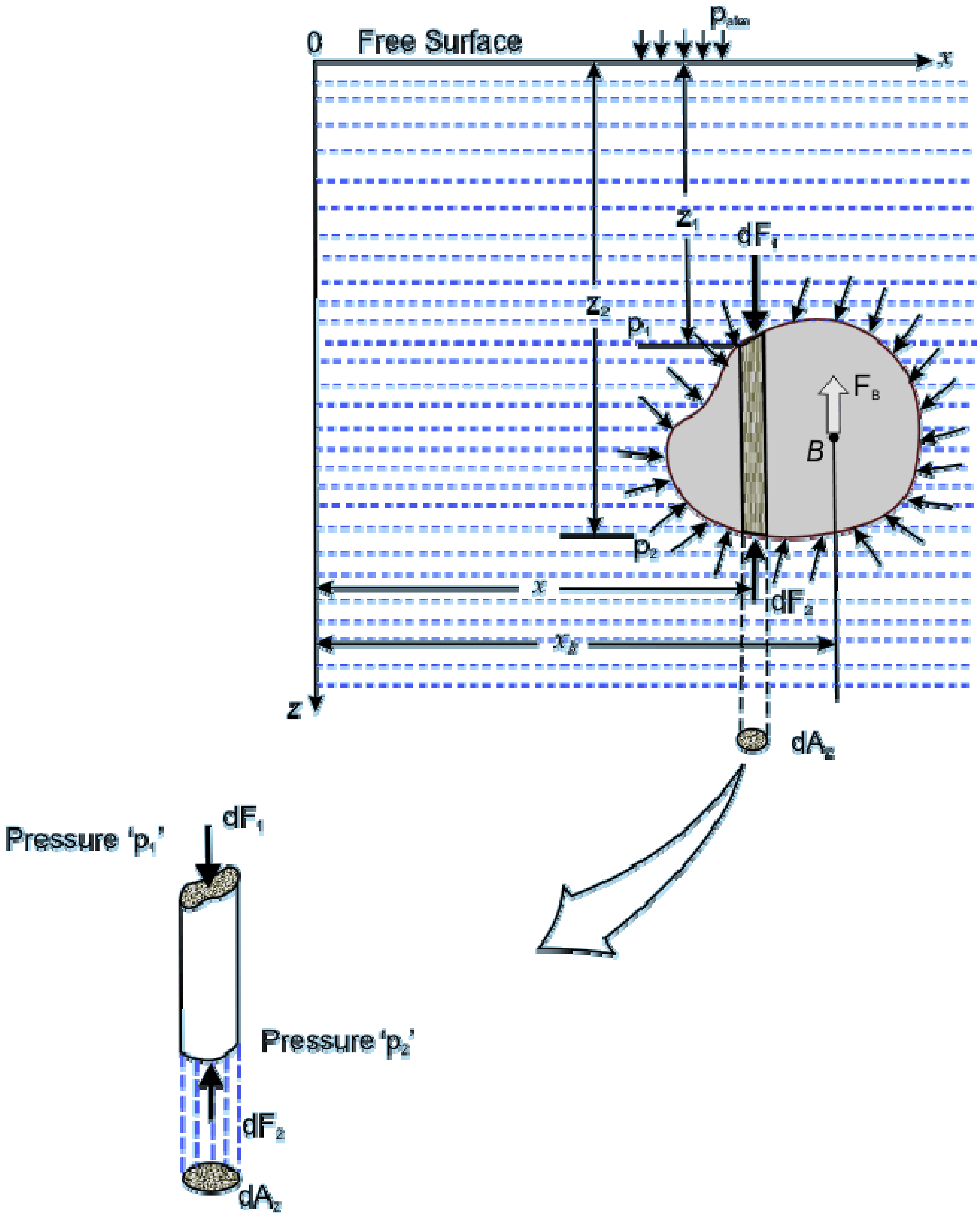


Fig 5.4 Buoyant Force on a Submerged Body

To calculate the vertical component of the resultant hydrostatic force, the body is considered to be divided into a number of elementary vertical prisms. The vertical forces acting on the two ends of such a prism of cross-section dA_z (Fig. 5.4) are respectively

$$dF_1 = (p_{atm} + p_1)dA_z = (p_{atm} + \rho g z_1)dA_z \quad (5.19a)$$

$$dF_2 = (p_{atm} + p_2)dA_z = (p_{atm} + \rho g z_2)dA_z \quad (5.19b)$$

Therefore, the buoyant force (the net vertically upward force) acting on the elemental prism of volume $d\forall$ is -

$$dF_B = dF_2 - dF_1 = \rho g(z_2 - z_1)dA_z = \rho g d\forall \quad (5.19c)$$

Hence the buoyant force F_B on the entire submerged body is obtained as

$$F_B = \iiint_{\forall} \rho g d\forall = \rho g \forall \quad (5.20)$$

Where $\forall$ **is the total volume of the submerged body.** The line of action of the force F_B can be found by taking moment of the force with respect to z-axis. Thus

$$x_B F_B = \int x dF_B \quad (5.21)$$

Substituting for dF_B and F_B from Eqs (5.19c) and (5.20) respectively into Eq. (5.21), the **x coordinate of the center of the buoyancy is** obtained as

$$x_B = \frac{1}{\forall}\iiint_{\forall} x d\forall \quad (5.22)$$

which is **the centroid of the displaced volume**. It is found from Eq. (5.20) that the buoyant force F_B equals to the weight of liquid displaced by the submerged body of volume $\forall$. This phenomenon was discovered by Archimedes and is known as the Archimedes principle.

ARCHIMEDES PRINCIPLE

The buoyant force on a submerged body

- The Archimedes principle states that the buoyant force on a submerged body is equal to the weight of liquid displaced by the body, and acts vertically upward through the centroid of the displaced volume.
- Thus the net weight of the submerged body, (the net vertical downward force experienced by it) is reduced from its actual weight by an amount that equals the buoyant force.

The buoyant force on a partially immersed body

- According to Archimedes principle, the buoyant force of a partially immersed body is equal to the weight of the displaced liquid.
- Therefore the buoyant force depends upon the density of the fluid and the submerged volume of the body.
- For a floating body in static equilibrium and in the absence of any other external force, the buoyant force must balance the weight of the body

3.8 Stability of Floating Bodies:

Stability of Unconstrained Submerged Bodies in Fluid

- The equilibrium of a body submerged in a liquid requires that the weight of the body acting through its centre of gravity should be colinear with equal hydrostatic lift acting through the centre of buoyancy.
- In general, if the body is not homogeneous in its distribution of mass over the entire volume, the location of **centre of gravity G does not coincide with the centre of volume, i.e., the centre of buoyancy B.**
- Depending upon the relative locations of G and B, a floating or submerged body attains three different states of equilibrium-

Let us suppose that a body is given a small angular displacement and then released. Then it will be said to be in

- Stable Equilibrium: If the body **returns to its original position** by retaining the originally vertical axis as vertical.
- Unstable Equilibrium: If the body **does not return to its original position but moves further** from it.
- Neutral Equilibrium: If the body **neither returns to its original position nor increases its displacement further**, it will simply adopt its new position.

Stable Equilibrium

Consider a submerged body in equilibrium whose centre of gravity is located below the centre of buoyancy (Fig. 5.5a). If the body is tilted slightly in any direction, the buoyant force and the weight always produce a restoring couple trying to return the body to its original position (Fig. 5.5b, 5.5c).

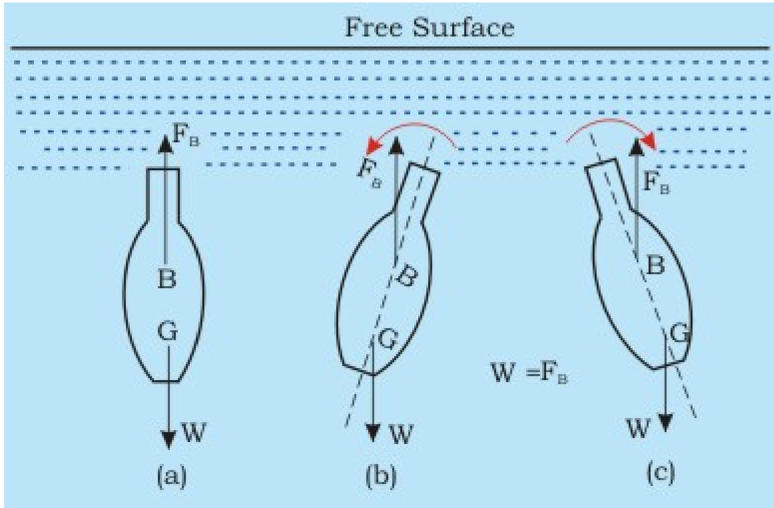


Fig 5.5 A Submerged body in Stable Equilibrium

Unstable Equilibrium

On the other hand, if point G is above point B (Fig. 5.6a), any disturbance from the equilibrium position will create a destroying couple which will turn the body away from its original position (5.6b, 5.6c).

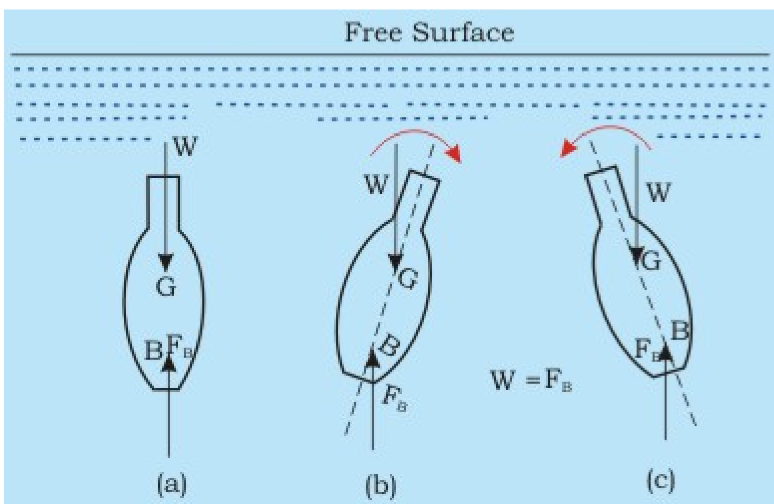


Fig 5.6 A Submerged body in Unstable Equilibrium

Neutral Equilibrium

When the centre of gravity G and centre of buoyancy B coincides, the body will always assume the same position in which it is placed (Fig 5.7) and hence it is in neutral equilibrium.

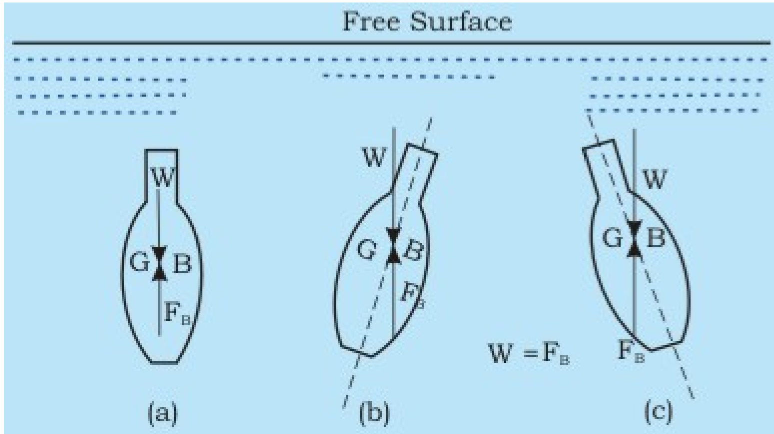


Fig 5.7 A Submerged body in Neutral Equilibrium

Therefore, it can be concluded that **a submerged body will be in stable, unstable or neutral equilibrium if its centre of gravity is below, above or coincident with the center of buoyancy respectively (Fig. 5.8).**

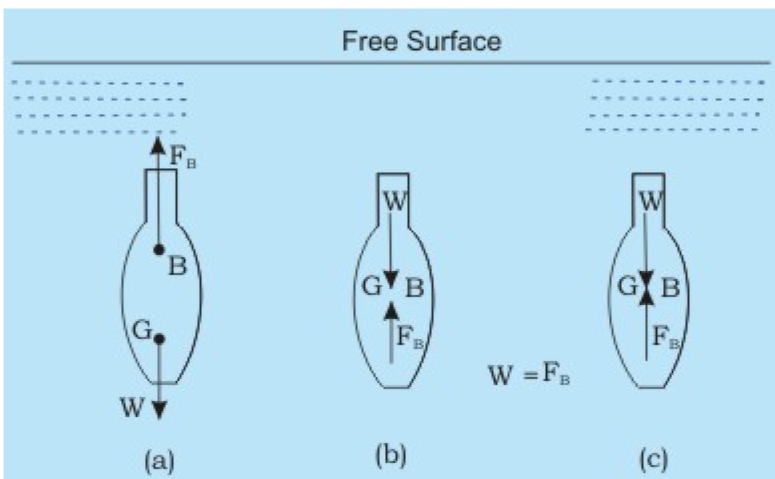


Fig 5.8 States of Equilibrium of a Submerged Body

(a) STABLE EQUILIBRIUM (B) UNSTABLE EQUILIBRIUM (C) NEUTRAL EQUILIBRIUM

Stability of Floating Bodies in Fluid

- When the body undergoes an angular displacement about a horizontal axis, the shape of the immersed volume changes and so the centre of buoyancy moves relative to the body.

- As a result of above observation stable equlibrium can be achieved, under certain condition, even when G is above B.
  Figure 5.9a illustrates a floating body -a boat, for example, in its equilibrium position.

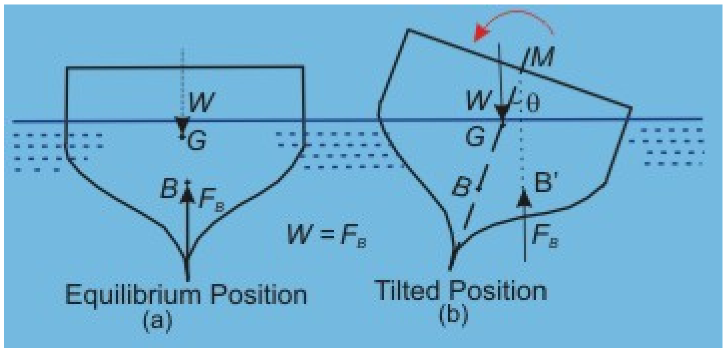


Fig 5.9 A Floating body in Stable equilibrium

Important points to note here are

a. The force of buoyancy F_B is equal to the weight of the body W

b. Centre of gravity G is above the centre of buoyancy in the same vertical line.

c. Figure 5.9b shows the situation after the body has undergone a small angular displacement θ with respect to the vertical axis.

d. The centre of gravity G remains unchanged relative to the body (This is not always true for ships where some of the cargo may shift during an angular displacement).

e. During the movement, the volume immersed on the right hand side increases while that on the left hand side decreases. Therefore the centre of buoyancy moves towards the right to its new position B'.

Let the new **line of action of the buoyant force** (which **is always vertical**) through B' intersects the axis BG (the old vertical line containing the centre of gravity G and the old centre of buoyancy B) at M. For small values of θ the **point M is** practically constant in position and is **known as metacentre**. For the body shown in Fig. 5.9, M is above G, and the couple acting on the body in its displaced position is a restoring couple which tends to turn the body to its original position. If M were below G, the couple would be an overturning couple and the original equilibrium would have been unstable. When M coincides with G, the body will assume its new position without any further movement and thus will be in neutral equilibrium. T**herefore, for a floating body, the stability is determined not simply by the relative position of B and G, rather by the relative position of M and G**. The distance of metacentre above G along the line BG is known as metacentric height GM which can be written as

$$GM = BM - BG$$

Hence the **condition of stable equilibrium for a floating body** can be expressed **in terms of metacentric height** as follows:

GM > 0 (M is above G) **Stable equilibrium**
GM = 0 (M coinciding with G) **Neutral equilibrium**
GM < 0 (M is below G) **Unstable equilibrium**

The angular displacement of a boat or ship about its longitudinal axis is known as 'rolling' while that about its transverse axis is known as "pitching".

Floating Bodies Containing Liquid

If a floating body carrying liquid with a free surface undergoes an angular displacement, the liquid will also move to keep its free surface horizontal. Thus not only does the centre of buoyancy B move, but also the centre of gravity G of the floating body and its contents move in the same direction as the movement of B. Hence the stability of the body is reduced. For this reason, liquid which has to be carried in a ship is put into a number of separate compartments so as to minimize its movement within the ship.

Period of Oscillation

The restoring couple caused by the buoyant force and gravity force acting on a floating body displaced from its equilibrium placed from its equilibrium position is $W.GM \sin\theta$ (Fig. 5.9). Since the torque equals to mass moment of inertia (i.e., second moment of mass) multiplied by angular acceleration, it can be written

$$W(GM)\sin\theta = -I_M \frac{d^2\theta}{dt^2} \tag{5.23}$$

Where I_M represents the mass moment of inertia of the body about its axis of rotation. The minus sign in the RHS of Eq. (5.23) arises since the torque is a retarding one and decreases the angular acceleration. If θ is small, sin θ=θ and hence Eq. (5.23) can be written as

$$\frac{d^2\theta}{dt^2} + \frac{W.GM}{I_M}\theta = 0 \tag{5.24}$$

Equation (5.24) represents a simple harmonic motion. The time period (i.e., the time of a complete oscillation from one side to the other and back again) equals to $2\pi(I_M/W.GM)^{1/2}$. The oscillation of the body results in a flow of the liquid around it and this flow has been disregarded here. In practice, of course, viscosity in the liquid introduces a damping action which quickly suppresses the oscillation unless further disturbances such as waves cause new angular displacements.

3.9 Centre of Gravity and Metacentric Height:

Refer to Section 3.8

Chapter 4

Description of Fluid Motion

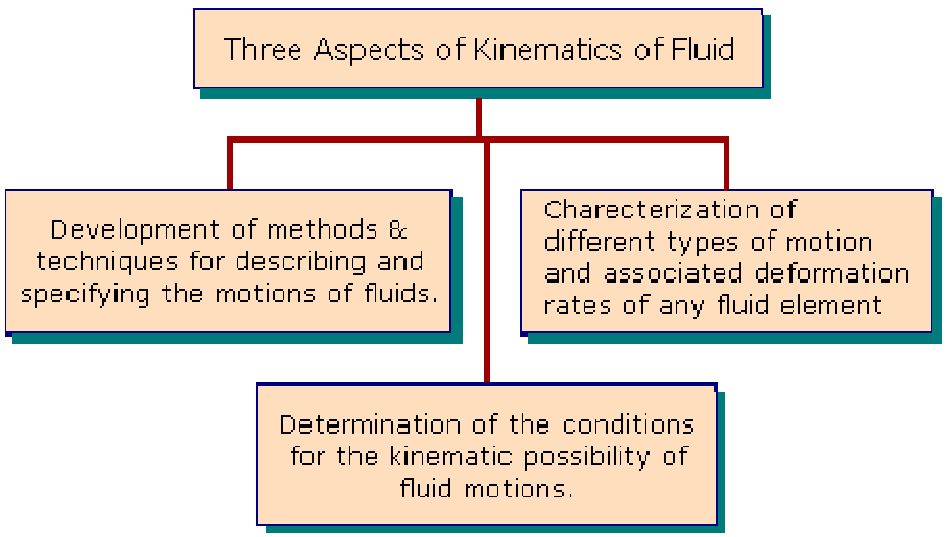
4.1 Lagrangian and Eulerian Methods:

Introduction

Kinematics is the geometry of Motion.

Kinematics of fluid describes the fluid motion and its consequences without consideration of the nature of forces causing the motion.

The subject has **three main aspects:**



Description of Fluid Motion

A. Lagrangian Method

- Using Lagrangian method, the fluid motion is described by tracing the **kinematic behaviour of each particl**e constituting the flow.
- Identities of the particles are made by specifying their initial position (spatial location) at a given time. The position of a particle at any other instant of time then becomes a function of its identity and time.

Analytical expression of the last statement :

$\vec{S} = S(\vec{S}_0, t)$ $\quad$ $\vec{S}$ is the position vector of a particle (with respect to a fixed point of reference) at a time t. (6.1)

$\vec{S}_0$ is its initial position at a given time t =t_0

Equation (6.1) can be written into scalar components with respect to a rectangular cartesian frame of coordinates as:

$x = x(x_0,y_0,z_0,t)$		(6.1a)
$y = y(x_0,y_0,z_0,t)$	(where, x_0,y_0,z_0 are the initial coordinates and x, y, z are the coordinates at a time t of the particle.)	(6.1b)
$z = z(x_0,y_0,z_0,t)$		(6.1c)

Hence $\vec{S}$ in can be expressed as

$\vec{S} = \vec{i}x + \vec{j}y + \vec{k}z$	$\vec{i}$, $\vec{j}$, and $\vec{k}$ are the unit vectors along x, y and z axes respectively.

velocity and acceleration

The **velocity** $\vec{V}$ and **acceleration** $\vec{a}$ of the fluid particle can be obtained from the material derivatives of the position of the particle with respect to time. Therefore,

$$\vec{V} = \left[\frac{d\vec{S}}{dt}\right]_{S_0} \qquad (6.2a)$$

In terms of scalar components,

$u = \left[\frac{dx}{dt}\right]_{x_0,y_0,z_0}$	(6.2b)
$v = \left[\frac{dy}{dt}\right]_{x_0,y_0,z_0}$	(6.2c)
$w = \left[\frac{dz}{dt}\right]_{x_0,y_0,z_0}$	(6.2d)

where **u, v, w are the components of velocity in x, y, z directions respectively.**

Similarly, for the **acceleration**,

$$\vec{a} = \left[\frac{d^2\vec{S}}{dt^2}\right]_{\vec{S}_o} \quad (6.3a)$$

and hence,

$$a_x = \left[\frac{d^2x}{dt^2}\right]_{x_o, y_o, z_o} \quad (6.3b)$$

$$a_y = \left[\frac{d^2y}{dt^2}\right]_{x_o, y_o, z_o} \quad (6.3c)$$

$$a_z = \left[\frac{d^2z}{dt^2}\right]_{x_o, y_o, z_o} \quad (6.3d)$$

where **a_x, a_y, a_z are accelerations in x, y, z directions respectively**.

Advantages of Lagrangian Method:

1. Since motion and trajectory of each fluid particle is known, its history can be traced.
2. Since particles are identified at the start and traced throughout their motion, conservation of mass is inherent.

Disadvantages of Lagrangian Method:

1. The solution of the equations presents appreciable mathematical difficulties except certain special cases and therefore, the method is rarely suitable for practical applications.

B. Eulerian Method

The method was developed by **Leonhard Euler**.

This method is of greater advantage since it avoids the determination of the movement of each individual fluid particle in all details.

It **seeks the velocity $\vec{V}$ and its variation with time t at each and every location ($\vec{S}$) in the flow field.**

In Eulerian view, all hydrodynamic parameters are functions of location and time.

Mathematical representation of the flow field in **Eulerian method**:

$$\vec{V} = V(\vec{S}, t) \tag{6.4}$$

where

$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$ and $\vec{S} = \vec{i}x + \vec{j}y + \vec{k}z$

Therefore,

$u = u\ (x, y, z, t)$
$v = v\ (x, y, z, t)$
$w = w\ (x, y, z, t)$

Relation between Eulerian and Lagrangian Method

The **Eulerian** description can be written as :

$$\frac{d\vec{S}}{dt} = V(\vec{S}, t) \tag{6.5}$$

or

$$\frac{dx}{dt} = u(x, y, z, t)$$

$$\frac{dy}{dt} = v(x, y, z, t)$$

$$\frac{dz}{dt} = w(x, y, z, t)$$

The integration of Eq. (6.5) yields the constants of integration which are to be found from the initial coordinates of the fluid particles.

Hence, the solution of Eq. (6.5) gives the equations of Lagrange as,

$$\vec{S} = S(\vec{S}_0, t)$$

or

$$x = x(x_0, y_0, z_0, t)$$

$$y = y(x_0, y_0, z_0, t)$$

$$z = z(x_0, y_0, z_0, t)$$

Above relation are same as **Lagrangian** formulation.

In principle, the **Lagrangian method of description can always be derived from the Eulerian method.**

<u>4.2 Description of Properties in a Moving Fluid:</u>

Scalar and Vector Fields

Scalar: Scalar is a quantity which can be expressed by a single number representing its **magnitude.**

Example: mass, density and temperature.

Scalar Field

If at every point in a region, a scalar function has a defined value, the region is called a **scalar field**.

Example: **Temperature distribution in a rod.**

Vector: Vector is a quantity which is specified by **both magnitude and direction**.

Example: Force, Velocity and Displacement.

Vector Field

If at every point in a region, a vector function has a defined value, the region is called a **vector field**.
Example: velocity field of a flowing fluid **.**

Flow Field

The region in which the flow parameters i.e. velocity, pressure etc. are defined at each and every point at any instant of time is called a **flow field**.

Thus a flow field would be specified by the velocities at different points in the region at different times.

Variation of Flow Parameters in Time and Space

Hydrodynamic parameters like pressure and density along with flow velocity may vary from one point to another and also from one instant to another at a fixed point.

According to **type of variations**, categorizing the flow:

Steady and Unsteady Flow

- **Steady Flow**
  A steady flow is defined as a flow in which the various hydrodynamic parameters and fluid properties at any point do not change with time.

In **Eulerian approach**, a steady flow is described as,

$$\vec{V} = V(\vec{S})$$

and

$$\vec{a} = a(\vec{S})$$

Implications:

1. Velocity and acceleration are functions of space coordinates only.

2. In a steady flow, the hydrodynamic parameters may vary with location, but the spatial distribution of these parameters remain invariant with time.

In the **Lagrangian approach**,

1. Time is inherent in describing the trajectory of any particle.
2. In steady flow, the velocities of all particles passing through any fixed point at different times will be same.
3. Describing velocity as a function of time for a given particle will show the velocities at different points through which the particle has passed providing the information of velocity as a function of spatial location as described by **Eulerian method**. Therefore, the Eulerian and Lagrangian approaches of describing fluid motion become identical under this situation.

- **Unsteady Flow**
  An unsteady Flow is defined as a flow in which the hydrodynamic parameters and fluid properties changes with time.

Uniform and Non-uniform Flows

- **Uniform Flow**

The flow is defined as uniform flow when in the flow field the **velocity and other hydrodynamic parameters do not change** f**rom point to point at any instant of time.**

For a uniform flow, the velocity is a function of time only, which can be expressed in Eulerian description as

$$\vec{V} = V(t)$$

Implication:

1. For a uniform flow, there will be no spatial distribution of hydrodynamic and other parameters.
2. **Any hydrodynamic parameter will have a unique value in the entire field**, irrespective of whether it

   changes with time − **unsteady uniform flow** OR

   does not change with time − **steady uniform flow.**

3. Thus ,steadiness of flow and uniformity of flow does not necessarily go together.

- **Non-Uniform Flow**

When the **velocity and other hydrodynamic parameters changes from one point to another** the flow is defined as **non-uniform**.

Important points:

1. For a non-uniform flow, the changes with position may be found either in the direction of flow or in directions perpendicular to it.

2.Non-uniformity in a direction perpendicular to the flow is always encountered near solid boundaries past which the fluid flows.

Reason: All fluids possess **viscosity** which reduces the relative velocity (of the fluid w.r.t. to the wall) to zero at a solid boundary. This is known as **no-slip condition.**

Four possible combinations

Type	Example
1. Steady Uniform flow	Flow at constant rate through a duct of uniform cross-section (The region close to the walls of the duct is disregarded)
2. Steady non-uniform flow	Flow at constant rate through a duct of non-uniform cross-section (tapering pipe)
3. Unsteady Uniform flow	Flow at varying rates through a long straight pipe of uniform cross-section. (Again the region close to the walls is ignored.)
4. Unsteady non-uniform flow	Flow at varying rates through a duct of non-uniform cross-section.

4.3 Local and Material Rate of Change:

Material Derivative and Acceleration

- **Let the position of a particle at any instant t in a flow field be given by the space coordinates (x, y, z) with respect to a rectangular cartesian frame of reference.**

- **The velocity components u, v, w of the particle along x, y and z directions respectively can then be written in Eulerian form as**

$$u = u\,(x, y, z, t)$$
$$v = v\,(x, y, z, t)$$
$$w = w\,(x, y, z, t)$$

- After an infinitesimal time interval t , let the particle move to a new position given by the coordinates $(x + \Delta x, y + \Delta y, z + \Delta z)$.
- Its velocity components at this new position be $u + \Delta u$, $v + \Delta v$ and $w + \Delta w$.
- Expression of velocity components in the **Taylor's series** form:

$$u + \Delta u = u(x, y, z, t) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z + \frac{\partial u}{\partial t}\Delta t + \text{higher order terms in } \Delta x, \Delta y, \Delta z \text{ and } \Delta t$$

$$v + \Delta v = v(x, y, z, t) + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \frac{\partial v}{\partial z}\Delta z + \frac{\partial v}{\partial t}\Delta t + \text{higher order terms in } \Delta x, \Delta y, \Delta z \text{ and } \Delta t$$

$$w + \Delta w = w(x, y, z, t) + \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z + \frac{\partial w}{\partial t}\Delta t + \text{higher order terms in } \Delta x, \Delta y, \Delta z \text{ and } \Delta t$$

The increment in space coordinates can be written as -

$$\Delta x = u\Delta t, \quad \Delta y = v\Delta t \quad \text{and} \quad \Delta z = w\Delta t$$

Substituting the values of $\Delta x, \Delta y, \Delta z$ in above equations, we have

$$\frac{\Delta u}{\Delta t} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$ etc

- In the limit $\Delta t \to 0$, the **equation** becomes

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad (7.1a)$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \qquad (7.1b)$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \qquad (7.1c)$$

- The above equations tell that the operator for **total differential** with respect to time, D/Dt in a **convective field** is related to the **partial differential** as:

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z} \qquad (7.2)$$

Explanation of equation 7.2 :

- The total differential D/Dt is known as the material or **substantial derivative** with respect to time.
- The **first term** ¶/¶t in the right hand side of is known as **temporal or local derivative** which expresses the rate of change with time, at a fixed position.
- The **last three terms** in the right hand side of are together known as **convective derivative** which represents the time rate of change due to change in position in the field.

Explanation of equation 7.1 (a, b, c):

- The terms in the left hand sides of Eqs (7.1a) to (7.1c) are defined as x, y and z components of **substantial or materia**l acceleration.
- The first terms in the right hand sides of Eqs (7.1a) to (7.1c) represent the respective **local or temporal** accelerations, while the other terms are **convective** accelerations.

Thus we can write,

$$a_x=\frac{Du}{Dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \qquad (7.2a)$$

$$a_y=\frac{Dv}{Dt}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \qquad (7.2b)$$

$$a_z=\frac{Dw}{Dt}=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} \qquad (7.2c)$$

(Material or substantial acceleration) = (temporal or local acceleration) + (convective acceleration)

Important points:

1. In a steady flow, the **temporal** acceleration is zero, since the velocity at any point is invariant with time.

2. In a uniform flow, on the other hand, the **convective** acceleration is zero, since the velocity components are not the functions of space coordinates.

3. In a steady and uniform flow, both the temporal and convective acceleration vanish and hence there exists no material acceleration.

Existence of the components of acceleration for different types of flow is shown in the table below.

Type of Flow	Material Acceleration	
	Temporal	Convective
1. Steady Uniform flow	0	0
2. Steady non-uniform flow	0	exists
3. Unsteady Uniform flow	exists	0
4. Unsteady non-uniform flow	exists	exists

- In vector form, Components of **Acceleration** in **Cylindrical Polar Coordinate System (r, q , z)**

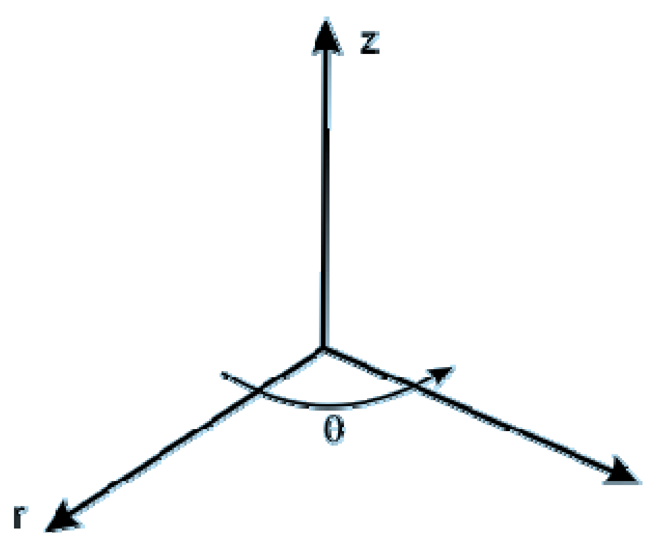


Fig 7.1 Velocity Components in a cylindrical Polar Coordinate System

- In a cylindrical polar coordinate system (Fig. 7.1), the components of acceleration in **r, θ and z** directions can be written as

$$a_r = \frac{DV_r}{Dt} - \frac{V_\theta^2}{r} = \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

$$a_\theta = \frac{DV_\theta}{Dt} + \frac{V_r V_\theta}{r} = \frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}$$

$$a_z = \frac{DV_z}{Dt} = \frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z}$$

Explanation of the additional terms appearing in the above equation:

1. The term $-\frac{V_\theta^2}{r}$ appears due to an inward radial acceleration arising from a change in the direction of V_θ (velocity component in the **azimuthal** direction) with θ as shown in Fig. 7.1. This is known as **centripetal** acceleration.
2. The term $V_r V_\theta / r$ represents a component of acceleration in **azimuthal direction** caused by **a change in the direction V_r of with θ**

4.4 Equations of Conservation of Mass for control Volume:

Conservation of Mass - The Continuity Equation

Law of conservation of mass

The law states that *mass can neither be created nor be destroyed*. Conservation of mass is inherent to a control mass system (closed system).

- The mathematical expression for the above law is stated as:

$$\Delta m/\Delta t = 0, \quad \text{where } m = \text{mass of the system}$$

- For a control volume (Fig.9.5), the principle of conservation of mass is stated as

Rate at which mass enters = Rate at which mass leaves the region + Rate of accumulation of mass in the region

OR

Rate of accumulation of mass in the control volume + Net rate of mass efflux from the control volume = 0 (9.1)

Continuity equation

The above statement expressed analytically in terms of velocity and density field of a flow is known as the **equation of continuity**.

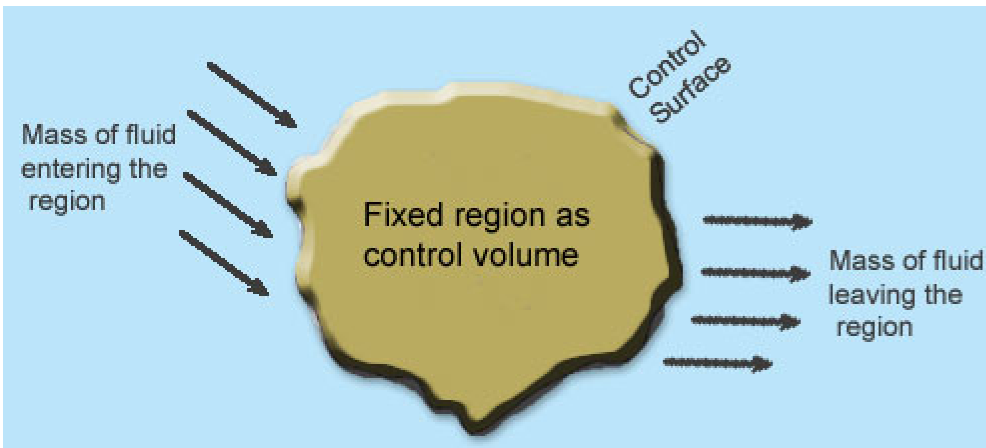


Fig 9.5 A Control Volume in a Flow Field

Continuity Equation - Differential Form

Derivation

1. The point at which the continuity equation has to be derived, is enclosed by an elementary control volume.
2. The influx, efflux and the rate of accumulation of mass is calculated across each surface within the control volume.

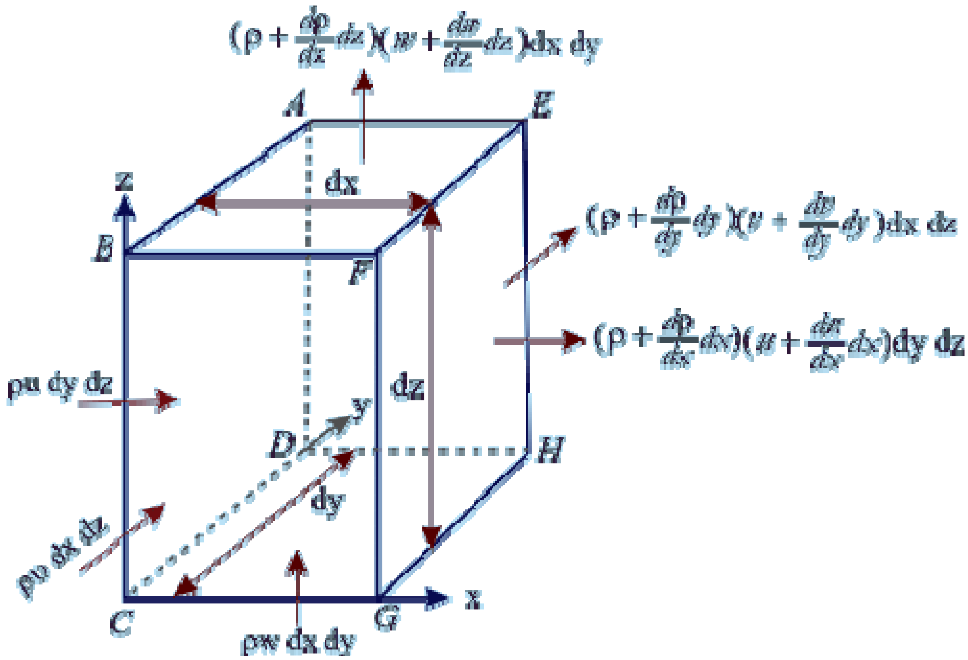


Fig 9.6 A Control Volume Appropriate to a Rectangular Cartesian Coordinate System

Consider a rectangular parallelopiped in the above figure as the control volume in a rectangular cartesian frame of coordinate axes.

- Net efflux of mass along x -axis must be the excess outflow over inflow across faces normal to x -axis.
- Let the fluid enter across one of such faces ABCD with a velocity u and a density ρ.The velocity and density with which the fluid will leave the face EFGH will be $u+\frac{\partial u}{\partial x}dx$ and $\rho+\frac{\partial \rho}{\partial x}dx$ respectively (neglecting the higher order terms in δx).
- Therefore, the rate of mass entering the control volume through face ABCD = $\rho u\, dy\, dz$.
- The rate of mass leaving the control volume through face EFGH will be

$$=\left(\rho+\frac{\partial \rho}{\partial x}dx\right)\left(u+\frac{\partial u}{\partial x}dx\right)dydz$$

$$=\left(\rho u+\frac{\partial}{\partial x}(\rho u)dx\right)dydz$$ (neglecting the higher order terms in dx)

- Similarly influx and efflux take place in all y and z directions also.
- Rate of accumulation for a point in a flow field

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}\rho(d\forall) = \frac{\partial \rho}{\partial t}d\forall$$

- Using, Rate of influx = Rate of Accumulation + Rate of Efflux

$$\rho u\,dy\,dz + \rho v\,dx\,dz + \rho w\,dx\,dy = \frac{\partial \rho}{\partial t}d\forall + \left(\rho + \frac{\partial \rho}{\partial x}dx\right)\left(u + \frac{\partial u}{\partial x}dx\right)dy\,dz$$
$$+\left(\rho + \frac{\partial \rho}{\partial y}dy\right)\left(v + \frac{\partial v}{\partial y}dy\right)dx\,dz + \left(\rho + \frac{\partial \rho}{\partial z}dz\right)\left(w + \frac{\partial w}{\partial z}dz\right)dx\,dy$$

- Transferring everything to right side

$$0 = \left[\left(\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x}\right) + \left(\rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y}\right) + \left(\rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z}\right)\right]dx\,dy\,dz + \left(\frac{\partial \rho}{\partial t}\right)d\forall$$

$$\Rightarrow \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]d\forall = 0 \qquad (9.2)$$

This is the Equation of Continuity for a compressible fluid in a rectangular cartesian coordinate system.

Continuity Equation - Vector Form

- The continuity equation can be written in a vector form as

$$\frac{\partial \rho}{\partial t} + \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left[\rho u\hat{i} - \rho v\hat{j} + \rho w\hat{k}\right] = 0$$

or,

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{V}) = 0 \qquad (9.3)$$

where $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ is the velocity of the point

- In case of a **steady flow**,

$$\frac{\partial \rho}{\partial t} = 0$$

- Hence Eq. (9.3) becomes

$$\nabla.(\rho\vec{V}) = 0 \qquad (9.4)$$

- In a rectangular cartesian coordinate system

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (9.5)$$

- Equation (9.4) or (9.5) represents the **continuity equation for a steady flow**.

- In case of an incompressible flow,
  ρ = constant

- Hence,

$$\frac{\partial \rho}{\partial t} = 0$$

- Moreover

$$\nabla.(\rho\vec{V}) - \rho\nabla.(\vec{V})$$

- Therefore, the **continuity equation for an incompressible flow** becomes

$$\nabla.(\vec{V}) = 0 \qquad (9.6)$$

$$\text{or, } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (9.7)$$

- In cylindrical polar coordinates eq.9.7 reduces to

$$\frac{1}{R}\frac{\partial}{\partial R}(R^2 V_R)+\frac{1}{\sin\varphi}\frac{\partial V_\theta}{\partial \theta}+\frac{1}{\sin\varphi}\frac{\partial(V_\varphi \sin\varphi)}{\partial \varphi}=0$$

- Eq. (9.7) can be written in terms of the strain rate components as

$$\dot{\varepsilon}_{xx}+\dot{\varepsilon}_{yy}+\dot{\varepsilon}_{zz}=0 \qquad (9.8)$$

4.5 Streamlines, Pathlines and Streakline:

Streamlines

Definition: Streamlines are the Geometrical representation of the of the flow velocity.

Description:

- In the **Eulerian** method, the velocity vector is defined as a function of time and space coordinates.
- If for a fixed instant of time, a **space curve** is drawn so that it is **tangent** everywhere to the **velocity** vector, then this curve is called a **Streamline**.

Therefore, the Eulerian method gives a series of instantaneous streamlines of the state of motion (Fig. 7.2a).

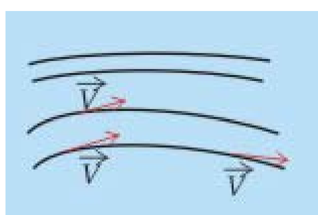


Fig 7.2a Streamlines

Alternative Definition: A streamline at any instant can be defined as an imaginary curve or line in the flow field so that the tangent to the curve at any point represents the direction of the **instantaneous velocity** at that point.

Comments:

- In an **unsteady flow** where the velocity vector changes with time, the pattern of streamlines also **changes from instant to instant.**

- In a **steady flow**, the orientation or the pattern of streamlines will be **fixed**.

From the above definition of streamline, it can be written as

$$\vec{V} \times d\vec{S} = 0 \qquad (7.3)$$

Description of the terms:

1. $d\vec{S}$ is the length of an infinitesimal line segment along a streamline at a point .

2. $\vec{V}$ is the instantaneous velocity vector.

The above expression therefore represents the **differential equation of a streamline**. In a cartesian coordinate-system, representing

$$\vec{S} = \vec{i}dx + \vec{j}dy + \vec{k}dz \qquad \vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$$

the above equation (Equation 7.3) may be simplified as

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad (7.4)$$

Stream tube:

A bundle of neighboring streamlines may be imagined to form a passage through which the fluid flows. This passage is known as a **stream-tube**.

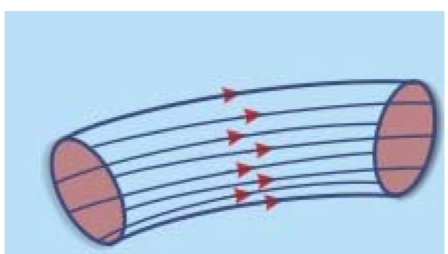


Fig 7.2b Stream Tube

Properties of Stream tube:

1. The stream-tube is bounded on all sides by streamlines.

2. Fluid velocity does not exist across a streamline, no fluid may enter or leave a stream-tube except through its ends.

3. The entire flow in a flow field may be imagined to be composed of flows through stream-tubes arranged in some arbitrary positions

Path Lines

Definition: A path line is the trajectory of a fluid particle of **fixed identity** as defined by Eq. (6.1).

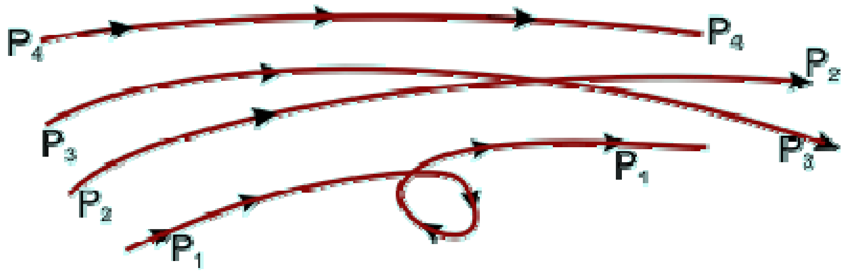


Fig 7.3 Path lines

A family of path lines represents the **trajectories of different particles**, say, P_1, P_2, P_3, etc. (Fig. 7.3).

Differences between Path Line and Stream Line

Path Line	Stream Line
▪ This refers to a path followed by a fluid particle over a period of time.	▪ This is an imaginary curve in a flow field for a fixed instant of time, tangent to which gives the instantaneous velocity at that point .
▪ Two path lines can intersect each other as or a single path line can form a loop as different particles or even same particle can arrive at the same point at different instants of time.	▪ Two stream lines can never intersect each other, as the instantaneous velocity vector at any given point is unique.

Note: In a steady flow **path lines** are **identical** to **streamlines** as the **Eulerian and Lagrangian** versions become the **same**.

Streak Lines

Definition: **A streak line** is the locus of the temporary locations of all particles that have passed though a fixed point in the flow field at any instant of time.

Features of a Streak Line:

- While a path line refers to the identity of a fluid particle, a streak line is specified by a fixed point in the flow field.
- It is of particular interest in experimental flow visualization.
- **Example:** If dye is injected into a liquid at a fixed point in the flow field, then at a later time t, the dye will indicate the end points of the path lines of particles which have passed through the injection point.
- The equation of a streak line at time t can be derived by the **Lagrangian method**.

If a fluid particle $(\vec{S}_o)$ passes through a fixed point $(\vec{S}_1)$ in course of time t, then the Lagrangian method of description gives the equation

$$S(\vec{S}_o, t) = \vec{S}_1 \qquad (7.5)$$

Solving for $\vec{S}_o$,

$$\vec{S}_o = F(\vec{S}_1, t) \qquad (7.6)$$

If the positions $(\vec{S}_o)$ of the particles which have passed through the fixed point $(\vec{S}_1)$ are determined, then **a streak line** can be drawn through these points.

Equation: The equation of the streak line at a time t is given by

$$\vec{S} = f(\vec{S}_o, t) \qquad (7.7)$$

Substituting Eq. (7.5) into Eq. (7.6) we get the **final form of equation of the streak line**,

$$\vec{S} = f\left[F(\vec{S}_1, t), t\right] \qquad (7.8)$$

Difference between Streak Line and Path Line

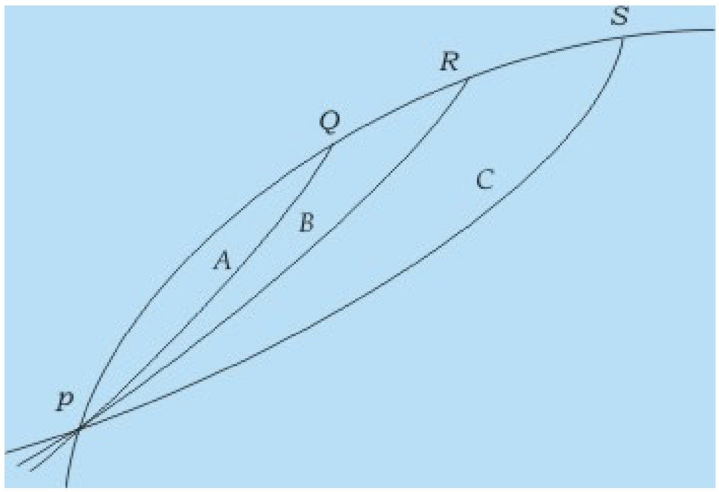


Fig 7.4 Description of a Streak line

Above diagram can be described by the following points:

Describing a **Path Line:**

a) Assume P be a fixed point in space through which particles of different identities pass at different times.

b) In an **unsteady flow**, the velocity vector at P will change with time and hence the particles arriving at P at **different times** will traverse

different paths like PAQ, PBR and PCS which represent the path lines of the particle.

Describing a **Streak Line:**

a) Let at any instant these particles arrive at points Q, R and S.

b) Q, R and S represent the end points of the trajectories of these three particles at the instant.

c) The curve joining the points S, R, Q and the fixed point P will define the **streak line** at that instant.

d) The fixed point P will also lie on the line, since at any instant, there will be always a particle of some identity at that point.

Above points show the differences.

Similarities:

a) For a **steady flow**, the velocity vector at any point is invariant with time

b) The path lines of the particles with different identities passing through P at different times will not differ

c) The path line would coincide with one another in a single curve which will indicate the streak line too.

Conclusion: Therefore, in a **steady flow**, the path lines, streak lines and streamlines are identical.

4.6 Vorticity and Circulation:

Translation of a Fluid Element

The movement of a fluid element in space has three distinct features simultaneously.

- Translation
- Rate of deformation
- Rotation.

Figure 7.4 shows the picture of a pure translation in absence of rotation and deformation of a fluid element in a two-dimensional flow described by a rectangular cartesian coordinate system.

In absence of deformation and rotation,

a) There will be no change in the length of the sides of the fluid element.

b) There will be no change in the included angles made by the sides of the fluid element.

c) The sides are displaced in parallel direction.

This is possible when the flow velocities u (the x component velocity) and v (the y component velocity) are neither a function of x nor of y, i.e., the flow field is totally **uniform**.

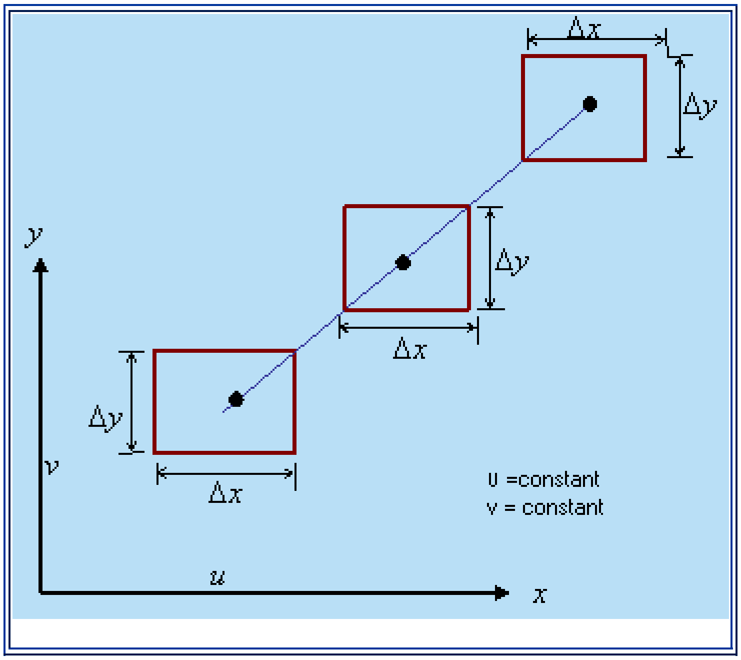


Fig 8.1 Fluid Element in pure translation

If a component of flow velocity becomes the function of only **one space coordinate** along which that velocity component is defined.

For example,

- if *u = u(x)* and *v = v(y),* the fluid element ABCD suffers a change in its linear dimensions along with translation
- there is no change in the included angle by the sides as shown in Fig. 7.5.

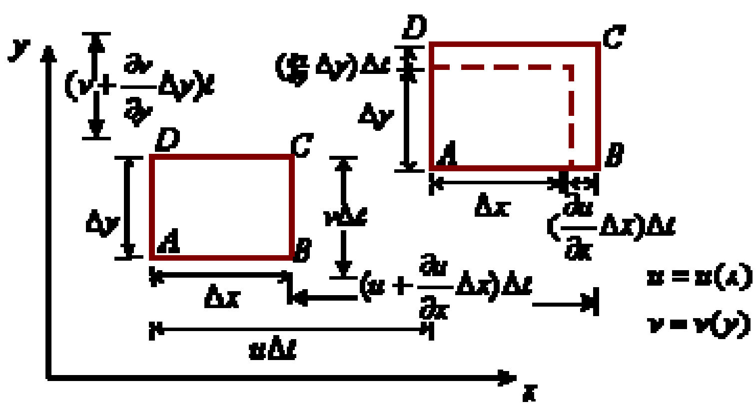


Fig 8.2 Fluid Element in Translation with Continuous Linear Deformation

The relative displacement of point B with respect to point A per unit time in x direction is

$$\frac{\partial u}{\partial x}\Delta x$$

Similarly, the relative displacement of D with respect to A per unit time in y direction is

$$\frac{\partial v}{\partial y}\Delta y$$

Translation with Linear Deformations

Observations from the figure:

Since u is not a function of y and *v* is not a function of x

- All points on the linear element AD move with same velocity in the x direction.
- All points on the linear element AB move with the same velocity in y direction.
- Hence the sides move parallel from their initial position without changing the included angle.

This situation is referred to as **translation with linear deformation**.

Strain rate:

The changes in lengths along the coordinate axes per unit time per unit original lengths are defined as the **components of linear deformation or strain rate in the respective directions**.

Therefore, **linear strain rate** component in the x direction

$$\dot{\varepsilon}_{xx} = \frac{\partial u}{\partial x}$$

and, linear strain rate component in y direction

$$\dot{\varepsilon}_{yy} = \frac{\partial v}{\partial y}$$

Rate of Deformation in the Fluid Element

Let us consider both the velocity component u and v are functions of x and y, i.e.,

$$u = u(x,y)$$

$v = v(x,y)$

Figure 8.3 represent the above condition

Observations from the figure:

- Point B has a relative displacement in y direction with respect to the point A.
- Point D has a relative displacement in x direction with respect to point A.
- The included angle between AB and AD changes.
- The fluid element suffers a continuous angular deformation along with the linear deformations in course of its motion.

Rate of Angular deformation:

The rate of angular deformation is defined as the **rate of change of angle** between the linear segments AB and AD which were initially perpendicular to each other.

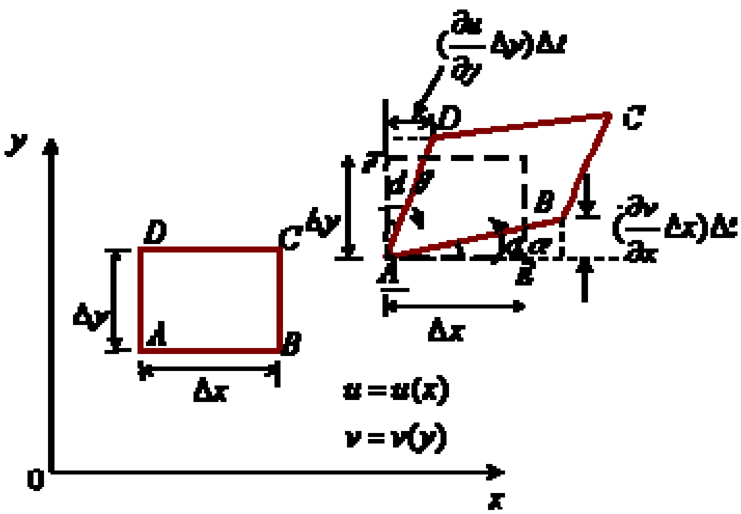


Fig 8.3 Fluid element in translation with simultaneous linear and angular deformation rates

From the above figure rate of angular deformation,

$$\dot{\gamma}_{xy} = \left(\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right) \qquad (8.1)$$

From the geometry

$$d\alpha = \frac{\partial v}{\partial x} dt \tag{8.2a}$$

$$d\alpha = \lim_{\Delta t \to 0} \left(\frac{\frac{\partial v}{\partial x} \Delta x \, \Delta t}{\Delta x \left(1 + \frac{\partial u}{\partial x} \Delta t \right)} \right) = \frac{\partial v}{\partial x} dt$$

$$d\beta = \lim_{\Delta t \to 0} \left(\frac{\frac{\partial u}{\partial y} \Delta y \, \Delta t}{\Delta y \left(1 + \frac{\partial v}{\partial y} \Delta t \right)} \right) = \frac{\partial u}{\partial y} dt$$

$$d\beta = \frac{\partial u}{\partial y} dt \tag{8.2b}$$

Hence,

$$\frac{d\alpha}{dt} + \frac{d\beta}{dt} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \tag{8.3}$$

Finally

$$\dot{\gamma}_{xy} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \tag{8.4}$$

Rotation

Figure 8.3 represent the situation of rotation

Observations from the figure:

- The transverse displacement of B with respect to A and the lateral displacement of D with respect to A (Fig. 8.3) can be considered as the rotations of the linear segments AB and AD about A.
- This brings the concept of rotation in a flow field.

Definition of rotation at a point:

The rotation at a point is defined as the **arithmetic mean** of the **angular velocities** of two perpendicular linear segments meeting at that point.

Example: The angular velocities of AB and AD about A are

$\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ respectively.

Considering the **anticlockwise direction as positive**, the rotation at A can be written as,

$$\omega_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right) \tag{8.5a}$$

or

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{8.5b}$$

The suffix z in ω represents the rotation about z-axis.

When $u = u\ (x, y)$ and $v = v\ (x, y)$ the **rotation and angular deformation of a fluid element exist simultaneously**.

Special case : Situation of pure Rotation

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad \dot{\gamma}_z = 0 \quad \text{and} \quad \omega_z = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Observation:

- The linear segments AB and AD move with the same angular velocity (both in magnitude and direction).
- The included angle between them remains the same and no angular deformation takes place. This situation is known as **pure rotation.**

Vorticity

Definition: The vorticity Ω in its simplest form is defined as a vector which is equal to **two times the rotation vector**

$$\vec{\Omega} = 2\vec{\omega} = \nabla \times \vec{V} \qquad (8.6)$$

For an **irrotational** flow, vorticity components are zero.

Vortex line:

If tangent to an imaginary line at a point lying on it is in the direction of the Vorticity vector at that point , the line is a **vortex line**.

The **general equation** of the **vortex line** can be written as,

$$\vec{\Omega} \times d\vec{r} = 0 \qquad (8.6b)$$

In a rectangular cartesian cartesian coordinate system, it becomes

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \qquad (8.6c)$$

where,

$$\Omega_x = 2\omega_x$$

$$\Omega_y = 2\omega_y$$

$$\Omega_z = 2\omega_z$$

Vorticity components as vectors:

The vorticity is actually an **anti symmetric tensor** and its three distinct elements transform like the components of a vector in cartesian coordinates.

This is the reason for which the vorticity components can be treated as vectors.

Existence of Flow

- A fluid must obey the law of conservation of mass in course of its flow as it is a material body.
- For a Velocity field to exist in a fluid continuum, the velocity components must obey the **mass conservation principle**.
- Velocity components which follow the mass conservation principle are said to constitute a possible fluid flow
- Velocity components violating this principle, are said to describe an impossible flow.
- The existence of a physically possible flow field is verified from the principle of conservation of mass.

The detailed discussion on this is deferred to the next chapter along with the discussion on principles of **conservation of momentum and energy**.

4.7 Laws of Vortex Motion:

Bernoulli's Equation In Irrotational Flow

In the previous lecture (lecture 13) we have obtained Bernoulli's equation

$$\boxed{\frac{p}{\rho}+\frac{V^2}{2}+gz=C}$$

- This equation was obtained by integrating the Euler's equation (the equation of motion) with respect to a displacement **'ds'** along a streamline. Thus, the value of C in the above equation is constant only along a streamline and should essentially vary from streamline to streamline.
- The equation can be used to define relation between flow variables at point B on the streamline and at point A, along the same streamline. So, in order to apply this equation, one should have knowledge of velocity field beforehand. This is one of the limitations of application of Bernoulli's equation.

Irrotationality of flow field

Under some special condition, the constant C becomes invariant from streamline to streamline and the Bernoulli's equation is applicable with same value of C to the entire flow field. The typical condition is the irrotationality of flow field.

Proof:

Let us consider a steady two dimensional flow of an ideal fluid in a rectangular Cartesian coordinate system. The velocity field is given by

$$\vec{V} = \hat{i}u + \hat{j}v$$

hence the condition of irrotationality is

$$\nabla \times \vec{V} = \left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \qquad (14.1)$$

The steady state Euler's equation can be written as

$$\rho\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} = -\frac{\partial p}{\partial x} \qquad (14.2a)$$

$$\rho\left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right\} = -\frac{\partial p}{\partial y} - \rho g \qquad (14.2b)$$

We consider the y-axis to be vertical and directed positive upward. From the condition of irrotationality given by the Eq. (14.1), we substitute $\frac{\partial v}{\partial x}$ in place of $\frac{\partial u}{\partial y}$ in the Eq. 14.2a and $\frac{\partial u}{\partial y}$ in place of $\frac{\partial v}{\partial x}$ in the Eq. 14.2b. This results in

$$\left\{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right\} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (14.3a)$$

$$\left\{u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right\} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g \qquad (14.3b)$$

Now multiplying Eq.(14.3a) by 'dx' and Eq.(14.3b) by 'dy' and then adding these two equations we have

$$u\left\{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right\} + v\left\{\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right\} = -\frac{1}{\rho}\left\{\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy\right\} - g\,dy \qquad (14.4)$$

The Eq. (14.4) can be physically interpreted as the equation of conservation of energy for an arbitrary displacement

$d\vec{r} = \vec{i}dx + \vec{j}dy$. Since, u, v and p are functions of x and y, we can write

$$du = \frac{\partial u}{\partial x}dx - \frac{\partial u}{\partial y}dy \quad (14.5a)$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \quad (14.5b)$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy \quad (14.5c)$$

With the help of Eqs (14.5a), (14.5b), and (14.5c), the Eq. (14.4) can be written as

$$udu + vdv = -\frac{1}{\rho}dp - gdy$$

$$d\left\{\frac{u^2}{2}\right\} + d\left\{\frac{v^2}{2}\right\} = -\frac{1}{\rho}dp - gdy$$

$$d\left\{\frac{u^2 + v^2}{2}\right\} = -\frac{1}{\rho}dp - gdy$$

$$d\left\{\frac{V^2}{2}\right\} = -\frac{1}{\rho}dp - gdy \quad (14.6)$$

The integration of Eq. 14.6 results in

$$\int\frac{dp}{\rho} + \frac{V^2}{2} + gy = C \quad (14.7a)$$

For an **incompressible flow**,

$$\boxed{\frac{p}{\rho} + \frac{V^2}{2} + gy = C} \quad (14.7b)$$

The constant C in Eqs (14.7a) and (14.7b) has the same value in the entire flow field, since no restriction was made in the choice of dr which was considered as an arbitrary displacement in evaluating the work.

Note: In deriving Eq. (13.8) the displacement **ds** was considered along a streamline. Therefore, the total mechanical energy remains constant everywhere in an inviscid and irrotational flow, while it is constant only along a streamline for an inviscid but rotational flow.

The equation of motion for the flow of an inviscid fluid can be written in a vector form as

$$\boxed{\frac{D\vec{V}}{Dt} = -\frac{\nabla p}{\rho} + \vec{X}}$$

where $\vec{X}$ is the body force vector per unit mass

Plane Circular Vortex Flows

- Plane circular vortex flows are defined as flows where streamlines are concentric circles. Therefore, with respect to a polar coordinate system with the centre of the circles as the origin or pole, the velocity field can be described as

$$V_\theta \neq 0 \qquad\qquad V_r = 0$$

where V_θ and V_r are the tangential and radial component of velocity respectively.

- The equation of continuity for a two dimensional incompressible flow in a polar coordinate system is

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0$$

which for a plane circular vortex flow gives $\frac{\partial V_\theta}{\partial \theta} = 0$ i.e. V_θ is not a function of θ. Hence, V_θ is a function of r only.

- We can write for the variation of total mechanical energy with radius as

$$\frac{\partial H}{\partial r} = \frac{V_\theta}{g}\left(\frac{dV_\theta}{dr} + \frac{V_\theta}{r}\right) \qquad (14.8)$$

Free Vortex Flows

- Free vortex flows are the plane circular vortex flows where the total mechanical energy remains constant in the entire flow field. There is neither any addition nor any destruction of energy in the flow field.

- Therefore, the total mechanical energy does not vary from streamline to streamline. Hence from Eq. (14.8), we have,

$$\frac{\partial H}{\partial r} = \frac{V_\theta}{g}\left(\frac{dV_\theta}{dr} + \frac{V_\theta}{r}\right) = 0$$

or, $$\frac{1}{r}\left[\frac{d}{dr}(V_\theta r)\right] = 0 \qquad (14.9)$$

- Integration of Eq 14.9 gives

$$V_\theta = \frac{C}{r} \qquad (14.10)$$

- The Eq. (14.10) describes the velocity field in a free vortex flow, where C is a constant in the entire flow field. The vorticity in a polar coordinate system is defined by -

$$\Omega = \frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}$$

- In case of vortex flows, it can be written as

$$\Omega = \frac{dV_\theta}{dr} + \frac{V_\theta}{r}$$

- For a free vortex flow, described by Eq. (14.10),Ω becomes zero. Therefore we conclude that a free vortex flow is irrotational, and hence, it is also referred to as **irrotational vortex**.
- It has been shown before that the total mechanical energy remains same throughout in an irrotational flow field. Therefore, irrotationality is a direct consequence of the constancy of total mechanical energy in the entire flow field and vice versa.
- The interesting feature in a free vortex flow is that as $r \rightarrow 0,\ V_\theta \rightarrow \infty$[Eq. (14.10)]. It mathematically signifies a point of singularity at r = 0 which, in practice, is impossible. In fact, the definition of a free vortex flow cannot be extended as r = 0 is approached.
- In a real fluid, friction becomes dominant as **r→0** and so a fluid in this central region tends to rotate as a solid body. Therefore, the singularity at r = 0 does not render the

theory of irrotational vortex useless, since, in practical problems, our concern is with conditions away from the central core.

Pressure Distribution in a Free Vortex Flow

- Pressure distribution in a vortex flow is usually found out by integrating the equation of motion in the r direction. The equation of motion in the radial direction for a vortex flow can be written as

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{V_\theta^2}{r} - g\cos\theta \qquad (14.11)$$

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{V_\theta^2}{r} - g\frac{dz}{dr} \qquad (14.12)$$

- Integrating Eq. (14.12) with respect to dr, and considering the flow to be incompressible we have,

$$\frac{p}{\rho} = \int \frac{V_\theta^2}{r}\,dr - gz + A \qquad (14.13)$$

- For a free vortex flow,

$$V_\theta = \frac{C}{r}$$

- Hence Eq. 14.13 becomes

$$\frac{p}{\rho} = -\frac{C^2}{2r^2} - gz + A \qquad (14.14)$$

- If the pressure at some radius **$r = r_a$,** is known to be the atmospheric pressure p_{atm} then equation (14.14) can be written as

$$\frac{p - p_{atm}}{\rho} = \frac{C^2}{2}\left(\frac{1}{r_a^2} - \frac{1}{r^2}\right) - g(z - z_a)$$

$$= \frac{(V_\theta^2)_{r=r_a}}{2} - \frac{V_\theta^2}{2} - g(z - z_a) \qquad (14.15)$$

where z and z_a are the vertical elevations (measured from any arbitrary datum) at r and r_a.

- Equation (14.15) can also be derived by a straight forward application of Bernoulli's equation between any two points at $r = r_a$ and $r = r$.

- **In a free vortex flow total mechanical energy remains constant.** There is neither any energy interaction between an outside source and the flow, nor is there any dissipation of mechanical energy within the flow. The fluid rotates by virtue of some rotation previously imparted to it or because of some internal action.

- Some examples are a whirlpool in a river, the rotatory flow that often arises in a shallow vessel when liquid flows out through a hole in the bottom (as is often seen when water flows out from a bathtub or a wash basin), and flow in a centrifugal pump case just outside the impeller.

Cylindrical Free Vortex

- **A cylindrical free** vortex motion is conceived in a cylindrical coordinate system with axis z directing vertically upwards (Fig. 14.1), where at each horizontal cross-section, there exists a planar free vortex motion with tangential velocity given by Eq. (14.10).
- The total energy at any point remains constant and can be written as

$$\frac{p}{\rho}+\frac{C^2}{2r^2}+gz=H\,(Const.) \qquad (14.16)$$

- The pressure distribution along the radius can be found from Eq. (14.16) by considering z as constant; again, for any constant pressure p, values of z, determining a surface of equal pressure, can also be found from Eq. (14.16).

- If p is measured in gauge pressure, then the value of z, where p = 0 determines the free surface (Fig. 14.1), if one exists.

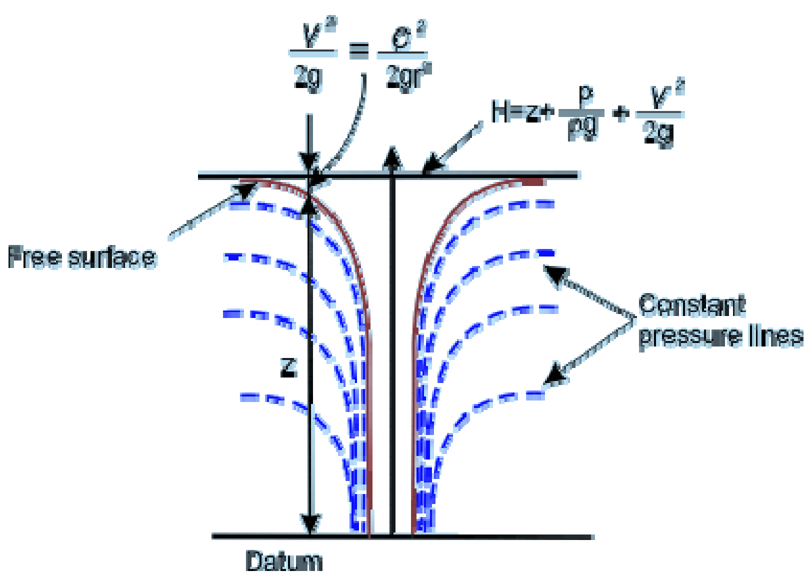


Fig 14.1 Cylindrical Free Vortex

Forced Vortex Flows

- Flows where streamlines are concentric circles and the tangential velocity is directly proportional to the radius of curvature are known as **plane circular forced vortex flows**.

- The flow field is described in a polar coordinate system as,

$$V_\theta = \omega r \qquad (14.17a)$$

and $$V_r = 0 \qquad (14.17b)$$

- All fluid particles rotate with the same angular velocity ω like a solid body. Hence a forced vortex flow is termed as a **solid body rotation**.

- The vorticity Ω for the flow field can be calculated as

$$\Omega = \frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}$$

$$= \omega - 0 + \omega = 2\omega$$

- Therefore, a forced vortex motion is not irrotational; rather it is a rotational flow with a constant vorticity 2ω. Equation (14.8) is used to determine the distribution of mechanical energy across the radius as

$$\frac{dH}{dr} = \frac{V_\theta}{g}\left(\frac{dV_\theta}{dr} + \frac{V_\theta}{r}\right) = \frac{2\omega^2 r}{g}$$

- Integrating the equation between the two radii on the same horizontal plane, we have,

$$H_2 - H_1 = \frac{\omega^2}{g}(r_2^2 - r_1^2) \qquad (14.18)$$

- Thus, we see from Eq. (14.18) that the total head (total energy per unit weight) increases with an increase in radius. The total mechanical energy at any point is the sum of kinetic energy, flow work or pressure energy, and the potential energy.

- Therefore the difference in total head between any two points in the same horizontal plane can be written as,

$$H_2 - H_1 = \left[\frac{p_2}{\rho g} - \frac{p_1}{\rho g}\right] + \left[\frac{V_2^2}{2g} - \frac{V_1^2}{2g}\right]$$

$$= \frac{p_2}{\rho g} - \frac{p_1}{\rho g} + \frac{\omega^2}{2g}(r_2^2 - r_1^2)$$

- Substituting this expression of H_2-H_1 in Eq. (14.18), we get

$$\frac{p_2 - p_1}{\rho} = \frac{\omega^2}{2}(r_2^2 - r_1^2)$$

- The same equation can also be obtained by integrating the equation of motion in a radial direction as

$$\int_1^2 \frac{1}{\rho}\frac{dp}{dr}dr = \int_1^2 \frac{V_\theta^2}{r}dr = \omega^2 \int_1^2 r\,dr$$

$$\frac{p_2 - p_1}{\rho} = \frac{\omega^2}{2}(r_2^2 - r_1^2)$$

- To maintain a forced vortex flow, mechanical energy has to be spent from outside and thus an external torque is always necessary to be applied continuously.

- Forced vortex can be generated by rotating a vessel containing a fluid so that the angular velocity is the same at all points.

4.8 Translation, Rotation and Rate of Deformation of a Fluid Particle:

Refer to Section 4.6

Chapter 5

Equations of Fluid Motion

5.1 Euler and Navier Stokes Equation:

Euler's Equation: The Equation of Motion of an Ideal Fluid

This section is not a mandatory requirement. One can skip this section (if he/she does not like to spend time on Euler's equation) and go directly to Steady Flow Energy Equation.

Using the Newton's second law of motion the relationship between the velocity and pressure field for a flow of an inviscid fluid can be derived. The resulting equation, in its differential form, is known as Euler's Equation. The equation is first derived by the scientist Euler.

Derivation:
Let us consider an elementary parallelopiped of fluid element as a control mass system in a frame of rectangular cartesian coordinate axes as shown in Fig. 12.3. The external forces acting on a fluid element are the body forces and the surface forces.

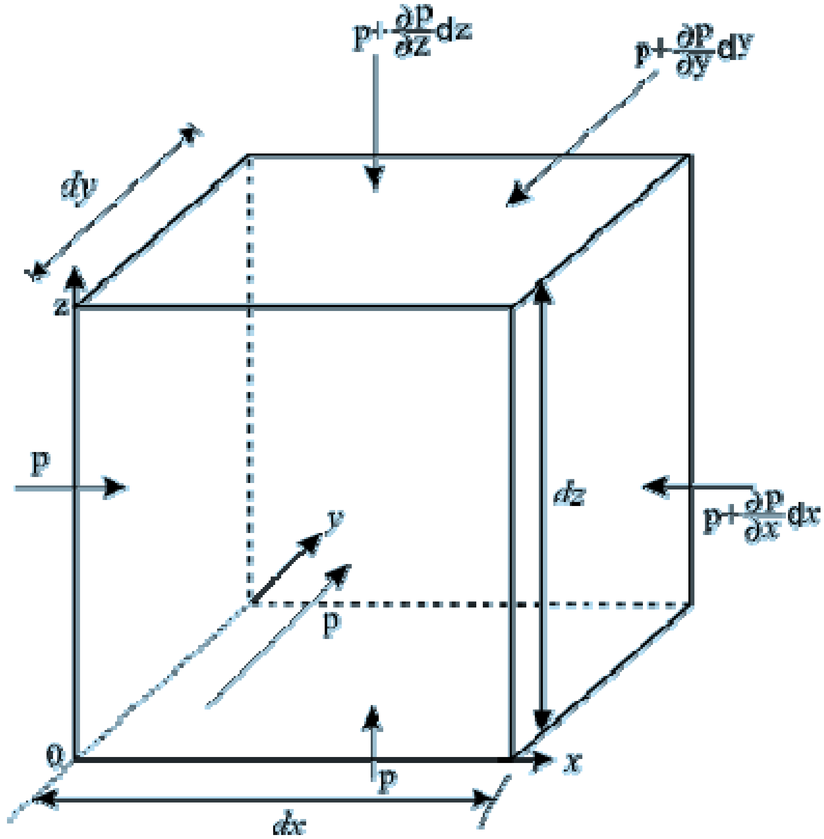


Fig 12.2 A Fluid Element appropriate to a Cartesian Coordinate System used for the derivation of Euler's Equation

Let X_x, X_y, X_z be the components of body forces acting per unit mass of the fluid element along the coordinate axes x, y and z respectively. The body forces arise due to external force fields like gravity, electromagnetic field, etc., and therefore, the detailed description of X_x, X_y and X_z are provided by the laws of physics describing the force fields. The surface forces for an inviscid fluid will be the pressure forces acting on different surfaces as shown in Fig. 12.3. Therefore, the net forces acting on the fluid element along x, y and z directions can be written as

$$F_x = \rho X_x dx\, dy\, dz + p\, dy\, dz - (p + \frac{\partial p}{\partial x} dx) dy\, dz = (\rho X_x - \frac{\partial p}{\partial x}) dx\, dy\, dz$$

$$F_y = \rho X_y dx\, dy\, dz + p\, dx\, dz - (p + \frac{\partial p}{\partial y} dy) dx\, dz = (\rho X_y - \frac{\partial p}{\partial y}) dx\, dy\, dz$$

$$F_z = \rho X_z dx\, dy\, dz + p\, dy\, dx - (p + \frac{\partial p}{\partial z} dz) dx\, dz = (\rho X_z - \frac{\partial p}{\partial z}) dx\, dy\, dz$$

Since each component of the force can be expressed as the rate of change of momentum in the respective directions, we have

$$\frac{D}{Dt}(\rho\, dx\, dy\, dz\, u) = \left(\rho X_x - \frac{\partial p}{\partial x}\right) dx dy dz \qquad (12.5a)$$

$$\frac{D}{Dt}(\rho\, dx\, dy\, dz\, v) = \left(\rho X_y - \frac{\partial p}{\partial y}\right) dx dy dz \qquad (12.5b)$$

$$\frac{D}{Dt}(\rho\, dx\, dy\, dz\, w) = \left(\rho X_z - \frac{\partial p}{\partial z}\right) dx dy dz \qquad (12.5c)$$

s the mass of a control mass system does not change with time, $\rho dx dy dz$ is constant with time and can be taken common. Therefore we can write Eqs (12.5a to 12.5c) as

$$\frac{Du}{Dt} = X_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (12.6a)$$

$$\frac{Dv}{Dt} = X_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (12.6b)$$

$$\frac{Dw}{Dt} = X_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (12.6c)$$

Expanding the material accelerations in Eqs (12.6a) to (12.6c) in terms of their respective temporal and convective components, we get

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (12.7a)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=X_y-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (12.7b)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=X_z-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (12.7c)$$

The Eqs (12.7a, 12.7b, 12.7c) are valid for both incompressible and compressible flow. By putting u = v = w = 0, as a special case, one can obtain the equation of hydrostatics . Equations (12.7a), (12.7b), (12.7c) can be put into a single vector form as

$$\frac{D\vec{V}}{Dt}=-\frac{\nabla p}{\rho}+\vec{X} \qquad (12.7d)$$

$$\frac{\partial \vec{V}}{\partial t}+(\vec{V}.\nabla)\vec{V}=\vec{X}-\frac{1}{\rho}\nabla p \qquad (12.7e)$$

where $\vec{V}$ the velocity vector and the body force vector per unit volume $\rho\vec{X}$ are defined as

$$\vec{V}=\hat{i}u+\hat{j}v+\hat{k}w$$

$$\rho\vec{X}=\hat{i}\rho X_x+\hat{j}\rho X_y+\hat{k}\rho X_z$$

Equation (12.7d) or (12.7e) is the well known **Euler's equation** in vector form, while Eqs (12.7a) to (12.7c) describe the Euler's equations in a rectangular Cartesian coordinate system.

Euler's Equation along a Streamline

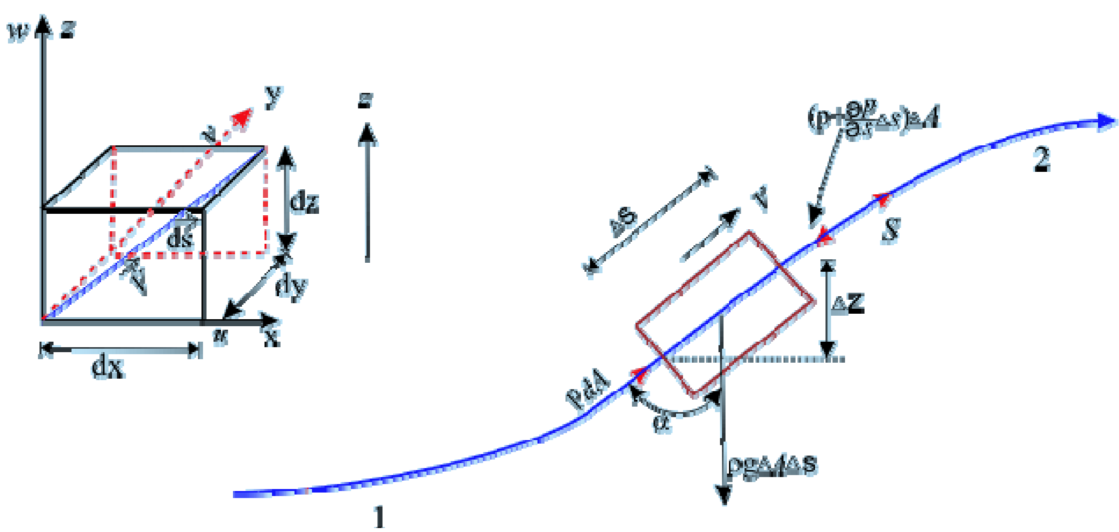


Fig 12.3 Force Balance on a Moving Element Along a Streamline

Derivation

Euler's equation along a streamline is derived by applying Newton's second law of motion to a fluid element moving along a streamline. Considering gravity as the only body force component acting vertically downward (Fig. 12.3), the net external force acting on the fluid element along the directions can be written as

$$F_s = -\frac{\partial p}{\partial s}\Delta s\Delta A - \rho\Delta s\Delta A g\cos\alpha \tag{12.8}$$

where ΔA is the cross-sectional area of the fluid element. By the application of Newton's second law of motion in s direction, we get

$$\rho\Delta s\Delta A\frac{DV}{Dt} = -\frac{\partial p}{\partial s}\Delta s\Delta A - \rho\Delta s\Delta A g\cos\alpha \tag{12.9}$$

From geometry we get

$$\cos\alpha = \lim_{\Delta s\to 0}\frac{\Delta z}{\Delta s} = \frac{dz}{ds}$$

Hence, the final form of Eq. (12.9) becomes

$$\rho\frac{DV}{Dt} = -\frac{\partial p}{\partial s} - \rho g\frac{dz}{ds}$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{dz}{ds} \tag{12.10}$$

Equation (12.10) is the Euler's equation along a streamline.

Let us consider $d\vec{s}$ along the streamline so that

$$d\vec{s} = \vec{i}dx + \vec{j}dy + \vec{k}dz$$

Again, we can write from Fig. 12.3

$$\frac{dx}{ds} = \frac{u}{V},\quad \frac{dy}{ds} = \frac{v}{V}\ \text{ and }\ \frac{dz}{ds} = \frac{w}{V}$$

The equation of a streamline is given by

$$\vec{V}\times d\vec{S} = 0$$

or, $$\begin{vmatrix} i & j & k \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0$$ which finally leads to

$$udy = vdx; \qquad udz = wdx; \qquad vdz = wdy$$

Multiplying Eqs (12.7a), (12.7b) and (12.7c) by dx, dy and dz respectively and then substituting the above mentioned equalities, we get

$$\rho\left(u\frac{\partial u}{\partial t}.\frac{ds}{V}+u\frac{\partial u}{\partial x}dx+u\frac{\partial u}{\partial y}dy+u\frac{\partial u}{\partial z}dz\right)=-\frac{\partial p}{\partial x}dx+X_x dx$$

$$\rho\left(v\frac{\partial v}{\partial t}.\frac{ds}{V}+v\frac{\partial v}{\partial x}dx+v\frac{\partial v}{\partial y}dy+v\frac{\partial v}{\partial z}dz\right)=-\frac{\partial p}{\partial y}dy+X_y dy$$

$$\rho\left(w\frac{\partial w}{\partial t}.\frac{ds}{V}+w\frac{\partial w}{\partial x}dx+w\frac{\partial w}{\partial y}dy+w\frac{\partial w}{\partial z}dz\right)=-\frac{\partial p}{\partial z}dy+X_z dz$$

Adding these three equations, we can write

$$\rho\left(\frac{ds}{V}.\frac{\partial}{\partial t}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)+\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)dx+\frac{\partial}{\partial y}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)dy+\frac{\partial}{\partial z}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)dz\right)$$

$$=\rho\left(\frac{ds}{V}.\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right)+\frac{\partial}{\partial x}\left(\frac{V^2}{2}\right)dx+\frac{\partial}{\partial y}\left(\frac{V^2}{2}\right)dy+\frac{\partial}{\partial z}\left(\frac{V^2}{2}\right)dz\right)$$

$$=\rho\left[\frac{\partial V}{\partial t}+V\left(\frac{\partial V}{\partial x}.\frac{dx}{ds}+\frac{\partial V}{\partial y}.\frac{dy}{ds}+\frac{\partial V}{\partial z}.\frac{dz}{ds}\right)\right]=-\left(\frac{\partial p}{\partial x}.\frac{dx}{ds}+\frac{\partial p}{\partial y}.\frac{dy}{ds}+\frac{\partial p}{\partial z}.\frac{dz}{ds}\right)-\rho g\frac{dz}{ds}$$

Hence, $$\boxed{\rho\left[\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial s}\right]=-\frac{\partial p}{\partial s}-\rho g\frac{dz}{ds}}$$

This is the more popular form of Euler's equation because the velocity vector in a flow field is always directed along the streamline.

Euler’s Equation in Different Conventional Coordinate System

Euler’s equation in different coordinate systems can be derived either by expanding the acceleration and pressure gradient terms of Eq. (12.7d), or by the application of Newton’s second law to a fluid element appropriate to the coordinate system.

Euler's Equation in Different Conventional Coordinate Systems

Coordinate System	Euler's Equation (Equation of motion for an inviscid flow)	
Rectangular Cartesian coordinate	x direction	$\frac{Du}{Dt} = X_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$
	y direction	$\frac{Dv}{Dt} = X_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$
	z direction	$\frac{Dw}{Dt} = X_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$
Cylindrical Polar Coordinate	r direction	$\frac{DV_r}{Dt} - \frac{V_\theta^2}{r} = X_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$
	θ direction	$\frac{DV_\theta}{Dt} - \frac{V_r V_\theta}{r} = X_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta}$
	z direction	$\frac{DV_z}{Dt} = X_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$
Spherical Polar Coordinate	R direction	$\frac{DV_R}{Dt} - \frac{V_\phi^2 + V_\theta^2}{R} = X_R - \frac{1}{\rho}\frac{\partial p}{\partial R}$
	θ direction	$\frac{DV_\theta}{Dt} + \frac{V_R V_\theta}{R} + \frac{V_\phi V_\theta \cot\phi}{R} = X_\theta - \frac{1}{R\sin\phi}\frac{\partial p}{\partial \theta}$
	φ direction	$\frac{DV_\phi}{Dt} + \frac{V_R V_\phi}{R} - \frac{V_\theta^2 \cot\phi}{R} = X_\phi - \frac{1}{\rho R}\frac{\partial p}{\partial \phi}$

A Control Volume Approach for the Derivation of Euler's Equation

Euler's equations of motion can also be derived by the use of the momentum theorem for a control volume.

Derivation

In a fixed x, y, z axes (the rectangular cartesian coordinate system), the parallelopiped which was taken earlier as a control mass system is now considered as a control volume (Fig. 12.4).

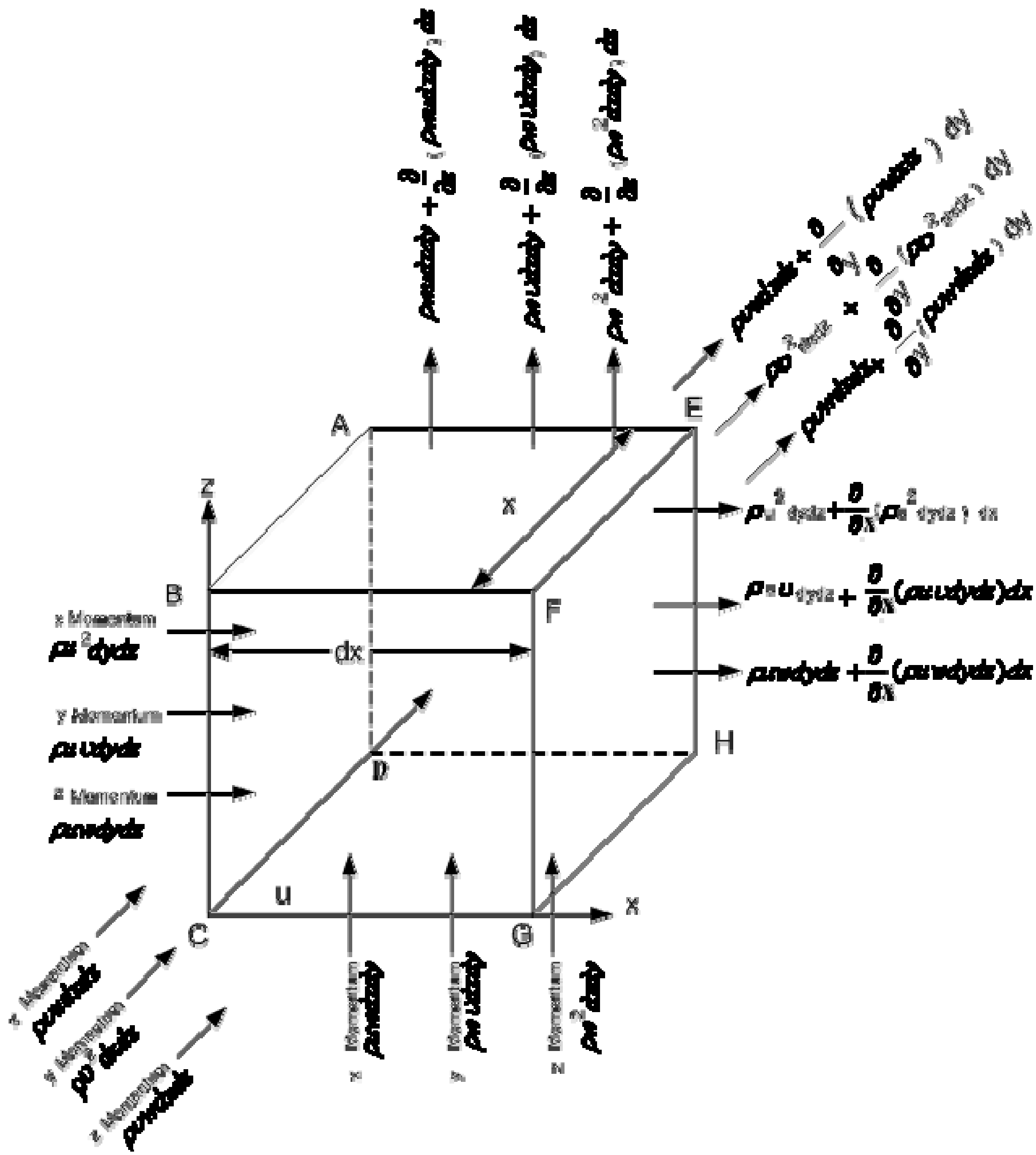


Fig 12.4 A Control Volume used for the derivation of Euler's Equation

We can define the velocity vector $\vec{V}$ and the body force per unit volume $\rho\vec{X}$ as

$$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$$

$$\rho \vec{X} = \hat{i}\rho X_x + \hat{j}\rho X_y + \hat{k}\rho X_z$$

The rate of x momentum influx to the control volume through the face ABCD is equal to ρu^2 dy dz. The rate of x momentum efflux from the control volume through the face EFGH equals

$$\rho u^2 dy\, dz + \frac{\partial}{\partial x}\left(\rho u^2 dy\, dz\right)dx$$

Therefore the rate of net efflux of x momentum from the control volume due to the faces perpendicular to the x direction (faces ABCD and EFGH) = $\frac{\partial}{\partial x}(\rho u^2)\mathrm{d}V$ where, $\mathrm{d}V$, the elemental volume = dx dy dz.

Similarly,

The rate of net efflux of x momentum due to the faces perpendicular to the y direction (face BCGF and ADHE) = $\frac{\partial}{\partial y}(\rho uv)\mathrm{d}V$

The rate of net efflux of x momentum due to the faces perpendicular to the z direction (faces DCGH and ABFE) = $\frac{\partial}{\partial z}(\rho uw)\mathrm{d}V$

Hence, the net rate of x momentum efflux from the control volume becomes

$$\left[\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial z}(\rho uv) + \frac{\partial}{\partial z}(\rho uw)\right]\mathrm{d}V$$

The time rate of increase in x momentum in the control volume can be written as

$\frac{\partial}{\partial t}(\rho u \mathrm{d}V) = \frac{\partial}{\partial t}(\rho u)\mathrm{d}V$ (Since, $\mathrm{d}V$, by the definition of control volume, is invariant with time) Applying the principle of momentum conservation to a control volume (Eq. 4.28b), we get

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = \rho X_x - \frac{\partial p}{\partial x} \qquad (12.11a)$$

The equations in other directions y and z can be obtained in a similar way by considering the y momentum and z momentum fluxes through the control volume as

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho vw) = \rho X_y - \frac{\partial p}{\partial y} \qquad (12.11b)$$

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial x}(\rho uw)+\frac{\partial}{\partial y}(\rho vw)+\frac{\partial}{\partial z}(\rho w^2)=\rho X_z-\frac{\partial p}{\partial z} \qquad (12.11c)$$

The typical form of Euler's equations given by Eqs (12.11a), (12.11b) and (12.11c) are known as the conservative forms.

Navier-Strokes Equation

- Generalized equations of motion of a real flow named after the inventors CLMH Navier and GG Stokes are derived from the **Newton's second law**
- **Newton's second law** states that the **product of mass and acceleration is equal to sum of the external forces acting on a body.**
- External forces are of two kinds-
    - one acts throughout the mass of the body ----- **body force** (gravitational force, electromagnetic force)
    - another acts on the boundary--------------------- **surface force** (pressure and frictional force).

Objective - We shall consider a differential fluid element in the flow field (Fig. 24.1). Evaluate the surface forces acting on the boundary of the rectangular parallelepiped shown below.

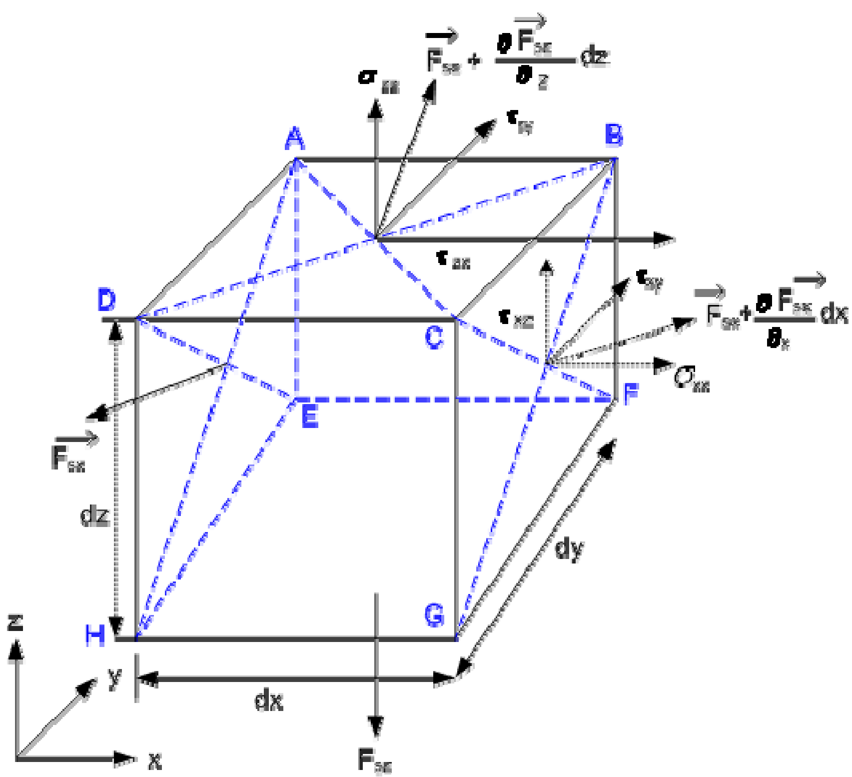


Fig. 24.1 Definition of the components of stress and their locations in a differential fluid element

- Let the body force per unit mass be

$$\vec{f}_b = \hat{i} f_x + \hat{j} f_y + \hat{k} f_z \qquad (24.6)$$

and surface force per unit volume be

$$\vec{F} = \hat{i} F_x + \hat{j} F_y + \hat{k} F_z \qquad (24.7)$$

- Consider surface force on the surface AEHD, per unit area,

$$\vec{F}_{sx} = \hat{i}\sigma_{xx} + \hat{j}\tau_{xy} + \hat{k}\tau_{xz}$$

[Here second subscript x denotes that the surface force is evaluated for the surface whose outward normal is the x axis]

- Surface force on the surface *BFGC* per unit area is

$$\vec{F}_{sx} + \frac{\partial \vec{F}_{sx}}{\partial x} dx$$

- Net force on the body due to imbalance of surface forces on the above two surfaces is

$$\frac{\partial \vec{F}_{sx}}{\partial x} dx\,dy\,dz \qquad (24.8)$$

(since area of faces AEHD and BFGC is dydz)

- Total force on the body due to net surface forces on all six surfaces is

$$\left(\frac{\partial \vec{F}_{sx}}{\partial x} + \frac{\partial \vec{F}_{sy}}{\partial y} + \frac{\partial \vec{F}_{sz}}{\partial z}\right) dx\,dy\,dz \qquad (24.9)$$

- And hence, the resultant surface force *dF*, per unit volume, is

$$d\vec{F} = \frac{\partial \vec{F}_{sx}}{\partial x} + \frac{\partial \vec{F}_{sy}}{\partial y} + \frac{\partial \vec{F}_{sz}}{\partial z} \qquad (24.10)$$

(since Volume= d*x* d*y* d*z*)

- The quantities $\vec{F}_{sx}$, $\vec{F}_{sy}$ and $\vec{F}_{sz}$ are vectors which can be resolved into normal stresses denoted by σ and shearing stresses denoted by τ as

$$\vec{F}_{sx} = \hat{i}\sigma_{xx} + \hat{j}\tau_{xy} + \hat{k}\tau_{xz} \qquad (24.11)$$

$$\vec{F}_y = \hat{i}\,\tau_{yx} + \hat{j}\,\sigma_{yy} + \hat{k}\,\tau_{yz}$$
$$\vec{F}_z = \hat{i}\,\tau_{zx} + \hat{j}\,\tau_{zy} + \hat{k}\,\sigma_{zz}$$

The **stress system has nine scalar quantities. These nine quantities form a stress tensor.**

Nine Scalar Quantities of Stress System - Stress Tensor

The set of nine components of stress tensor can be described as

$$\pi = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \qquad (24.12)$$

- The stress tensor is ***symmetric***,
- This means that two shearing stresses with subscripts which differ only in their sequence are equal. For example $\tau_{xz} = \tau_{zx}$

- Considering the equation of motion for instantaneous rotation of the fluid element (Fig. 24.1) about y axis, we can write

$$\dot{\omega}_y \, dI_y - (\tau_{xz}\, dy\, dz)dx - (\tau_{zx}\, dx\, dy)dz$$
$$= (\tau_{xz} - \tau_{zx})dV$$

where dV =dxdydz is the volume of the element, $\dot{\omega}_y$ is the angular acceleration

dI_y is the moment of inertia of the element about y-axis

- Since dI_y is proportional to fifth power of the linear dimensions and dV is proportional to the third power of the linear dimensions, the left hand side of the above equation and the second term on the right hand side vanishes faster than the first term on the right hand side on contracting the element to a point.
- Hence, the result is

$$\tau_{xz} = \tau_{zx}$$

From the similar considerations about other two remaining axes, we can write

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

which has already been observed in Eqs (24.2a), (24.2b) and (24.2c) earlier.

- Invoking these conditions into Eq. (24.12), the stress tensor becomes

$$\pi = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \tag{24.13}$$

- Combining Eqs (24.10), (24.11) and (24.13), the resultant surface force per unit volume becomes

$$\begin{aligned} d\vec{F} = {} & \hat{i}\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}\right) \\ & +\hat{j}\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}\right) \\ & +\hat{k}\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}\right) \end{aligned} \tag{8.14}$$

- As per the velocity field,

$$\frac{D\vec{V}}{Dt} = \hat{i}\frac{Du}{Dt}+\hat{j}\frac{Dv}{Dt}+\hat{k}\frac{Dw}{Dt} \tag{24.15}$$

By Newton's law of motion applied to the differential element, we can write

$$\rho(dx\,dy\,dz)\frac{D\vec{V}}{Dt} = \left(d\vec{F}\right)(dx\,dy\,dz)+\rho\vec{f}_b(dx\,dy\,dz)$$

or, $$\rho\frac{D\vec{V}}{Dt} = d\vec{F} + \rho\vec{f}_b$$

Substituting Eqs (24.15), (24.14) and (24.6) into the above expression, we obtain

$$\rho\frac{Du}{Dt} = \rho f_x + \left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}\right) \tag{24.16a}$$

$$\rho\frac{Dv}{Dt} = \rho f_y + \left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}\right) \tag{24.16b}$$

$$\rho \frac{Dw}{Dt} = \rho f_z + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \qquad (24.16c)$$

Since $\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu(\nabla \cdot \vec{V})$

$$\frac{\partial \sigma_{xx}}{\partial x} = -\partial p/\partial x + \partial/\partial x\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla \cdot \vec{V}\right)\right]$$

Similarly others follow.

- So we can express $\frac{Du}{Dt}$, $\frac{Dv}{Dt}$ and $\frac{Dw}{Dt}$ in terms of field derivatives,

$$\rho \frac{Du}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla \cdot \vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \qquad (24.17a)$$

$$\rho \frac{Dv}{Dt} = \rho f_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla \cdot \vec{V}\right)\right] - \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \qquad (24.17b)$$

$$\rho \frac{Dw}{Dt} = \rho f_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla \cdot \vec{V}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \qquad (24.17c)$$

- These differential equations are known as Navier-Stokes equations.
- At this juncture, discuss the equation of continuity as well, which has a **general (conservative) form**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad (24.18)$$

- In case of incompressible flow ρ = constant. Therefore, equation of continuity for incompressible flow becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (24.19)$$

- Invoking Eq. (24.19) into Eqs (24.17a), (24.17b) and (24.17c), we get

$$\rho\frac{Du}{Dt}=\rho f_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}\right)\right]+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\mu\left(\frac{\partial}{\partial y}\frac{\partial v}{\partial x}\right)+\mu\left(\frac{\partial}{\partial z}\frac{\partial w}{\partial x}\right)$$

$$=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^{0}$$

Similarly others follow

Thus,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad (24.20a)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho f_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad (24.20b)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho f_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \qquad (24.20c)$$

Vector Notation& derivation in Cylindrical Coordinates - Navier-Stokes equation

- Using, vector notation to write Navier-Stokes and continuity equations for incompressible flow we have

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{f}_b-\nabla p+\mu\nabla^2\vec{V} \qquad (24.21)$$

and

$$\nabla\cdot\vec{V}=0 \qquad (24.22)$$

- we have **four unknown quantities, u, v, w and p** ,
- we also have **four equations, - equations of motion in three directions and the continuity equation.**
- In principle, these equations are solvable but to date generalized solution is not available due to the complex nature of the set of these equations.
- The highest order terms, which come from the viscous forces, are linear and of second order
- The first order convective terms are non-linear and hence, the set is termed as **quasi-linear.**

- Navier-Stokes equations in cylindrical coordinate (Fig. 24.2) are useful in solving many problems. If , and denote the velocity components along the radial, cross-radial and axial directions respectively, then for the case of incompressible flow, Eqs (24.21) and (24.22) lead to the following system of equations:

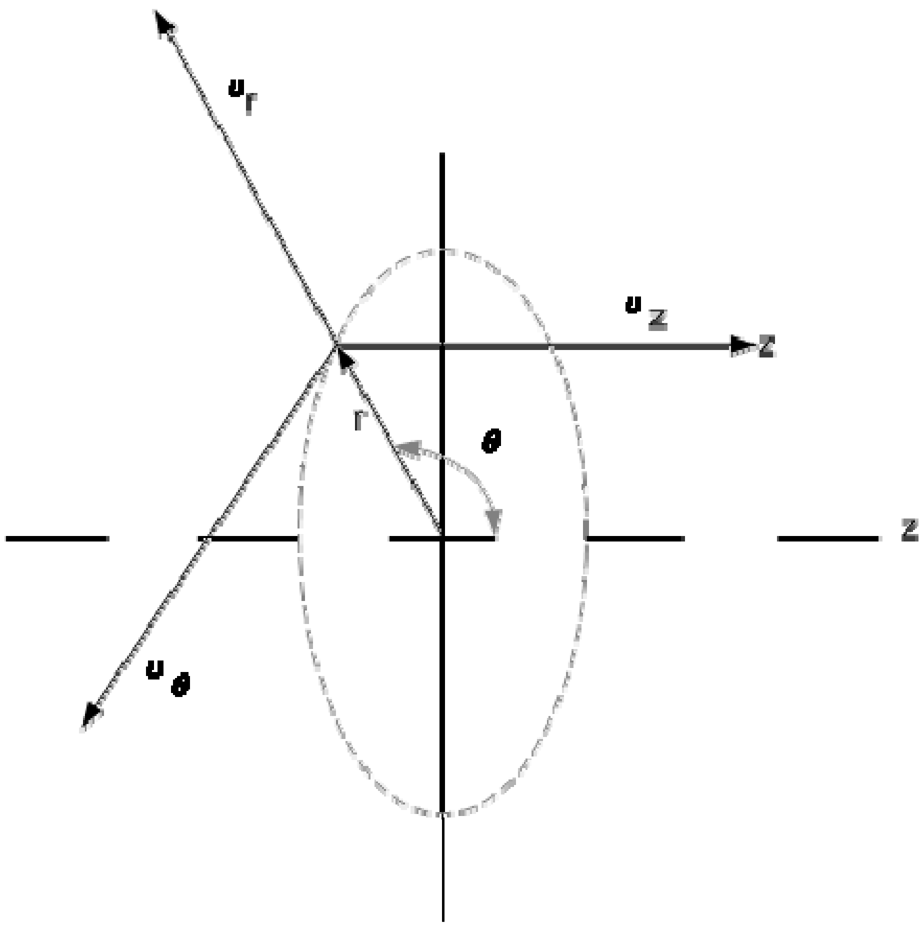


FIG 24.2 Cylindrical polar coordinate and the velocity components

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\cdot\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=\rho f_r-\frac{\partial p}{\partial r}$$

$$+\mu\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}+\frac{1}{r^2}\cdot\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\cdot\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right) \quad (24.23a)$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\cdot\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}\right)=\rho f_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$+\mu\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{1}{r^2}\cdot\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\cdot\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right) \quad (24.23b)$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\cdot\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=\rho f_z-\frac{\partial p}{\partial z}$$
$$+\mu\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right) \quad (24.23c)$$

$$\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0 \quad (24.24)$$

5.2 Derivation of Bernoulli's Equation for Inviscid and Viscous Flow Field

Bernoulli's Equation

Energy Equation of an ideal Flow along a Streamline

Euler's equation (the equation of motion of an inviscid fluid) along a stream line for a steady flow with gravity as the only body force can be written as

$$V\frac{dV}{ds}=-\frac{1}{\rho}\frac{dp}{ds}-g\frac{dz}{ds} \quad (13.6)$$

Application of a force through a distance ds along the streamline would physically imply work interaction. Therefore an equation for conservation of energy along a streamline can be obtained by integrating the Eq. (13.6) with respect to ds as

$$\int V\frac{dV}{ds}ds=-\int\frac{1}{\rho}\frac{dp}{ds}ds-\int g\frac{dz}{ds}ds$$
$$\text{or,}\quad \frac{V^2}{2}+\int\frac{dp}{\rho}+gz=C \quad (13.7)$$

Where C is a constant along a streamline. In case of an incompressible flow, Eq. (13.7) can be written as

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=C \quad (13.8)$$

The Eqs (13.7) and (13.8) are based on the assumption that no work or heat interaction between a fluid element and the surrounding takes place. The first term of the Eq. (13.8) represents the flow work per unit mass, the second term represents the kinetic energy per unit mass and the third term represents the potential energy per unit mass. Therefore the sum of three terms in the left hand side of Eq. (13.8) can be considered as the total mechanical energy per unit mass which remains constant along a streamline for a steady inviscid and incompressible flow of fluid. Hence the Eq. (13.8) is also known as **Mechanical energy equation**.

This equation was developed first by Daniel Bernoulli in 1738 and is therefore referred to as Bernoulli's equation. Each term in the Eq. (13.8) has the dimension of energy per unit mass. The equation can also be expressed in terms of energy per unit weight as

$$\frac{p}{\rho g}+\frac{V^2}{2g}+z=C_1\,(\text{constant}) \tag{13.9}$$

In a fluid flow, the energy per unit weight is termed as head. Accordingly, equation 13.9 can be interpreted as

Pressure head + Velocity head + Potential head =Total head (total energy per unit weight).

Bernoulli's Equation with Head Loss

The derivation of mechanical energy equation for a real fluid depends much on the information about the frictional work done by a moving fluid element and is excluded from the scope of the book. However, in many practical situations, problems related to real fluids can be analysed with the help of a modified form of Bernoulli's equation as

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2+h_f \tag{13.10}$$

where, h_f represents the frictional work done (the work done against the fluid friction) per unit weight of a fluid element while moving from a station 1 to 2 along a streamline in the direction of flow. The term h_f is usually referred to as head loss between 1 and 2, since it amounts to the loss in total mechanical energy per unit weight between points 1 and 2 on a streamline due to the effect of fluid friction or viscosity. It physically signifies that the difference in the total mechanical energy between stations 1 and 2 is dissipated into intermolecular or thermal energy and is expressed as loss of head h_f in Eq. (13.10). The term head loss, is conventionally symbolized as h_L instead of h_f in dealing with practical problems. For an inviscid flow $h_L = 0$, and the total mechanical energy is constant along a streamline.

Bernoulli's Equation In Irrotational Flow

In the previous lecture (lecture 13) we have obtained Bernoulli's equation

$$\boxed{\frac{p}{\rho}+\frac{V^2}{2}+gz=C}$$

- This equation was obtained by integrating the Euler's equation (the equation of motion) with respect to a displacement '**ds**' along a streamline. Thus, the value of C in the above equation is constant only along a streamline and should essentially vary from streamline to streamline.
- The equation can be used to define relation between flow variables at point B on the streamline and at point A, along the same streamline. So, in order to apply this equation,

one should have knowledge of velocity field beforehand. This is one of the limitations of application of Bernoulli's equation.

Irrotationality of flow field

Under some special condition, the constant C becomes invariant from streamline to streamline and the Bernoulli's equation is applicable with same value of C to the entire flow field. The typical condition is the irrotationality of flow field.

Click here to play the demonstration

Proof:

Let us consider a steady two dimensional flow of an ideal fluid in a rectangular Cartesian coordinate system. The velocity field is given by

$$\vec{V} = \vec{i}u + \vec{j}v$$

hence the condition of irrotationality is

$$\nabla \times \vec{V} = \left\{ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right\} = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \qquad (14.1)$$

The steady state Euler's equation can be written as

$$\rho\left\{ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right\} = -\frac{\partial p}{\partial x} \qquad (14.2a)$$

$$\rho\left\{ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right\} = -\frac{\partial p}{\partial y} - \rho g \qquad (14.2b)$$

We consider the y-axis to be vertical and directed positive upward. From the condition of irrotationality given by the Eq. (14.1), we substitute $\frac{\partial v}{\partial x}$ in place of $\frac{\partial u}{\partial y}$ in the Eq. 14.2a and $\frac{\partial u}{\partial y}$ in place of $\frac{\partial v}{\partial x}$ in the Eq. 14.2b. This results in

$$\left\{ u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} \right\} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (14.3a)$$

$$\left\{ u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} \right\} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g \qquad (14.3b)$$

Now multiplying Eq.(14.3a) by 'dx' and Eq.(14.3b) by 'dy' and then adding these two equations we have

$$u\left\{\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right\}+v\left\{\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right\}=-\frac{1}{\rho}\left\{\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy\right\}-gdy \qquad (14.4)$$

The Eq. (14.4) can be physically interpreted as the equation of conservation of energy for an arbitrary displacement

$d\vec{r}=\vec{i}dx+\vec{j}dy$. Since, u, v and p are functions of x and y, we can write

$$du=\frac{\partial u}{\partial x}dx-\frac{\partial u}{\partial y}dy \qquad (14.5a)$$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy \qquad (14.5b)$$

$$dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy \qquad (14.5c)$$

With the help of Eqs (14.5a), (14.5b), and (14.5c), the Eq. (14.4) can be written as

$$udu+vdv=-\frac{1}{\rho}dp-gdy$$

$$d\left\{\frac{u^2}{2}\right\}+d\left\{\frac{v^2}{2}\right\}=-\frac{1}{\rho}dp-gdy$$

$$d\left\{\frac{u^2+v^2}{2}\right\}=-\frac{1}{\rho}dp-gdy$$

$$d\left\{\frac{V^2}{2}\right\}=-\frac{1}{\rho}dp-gdy \qquad (14.6)$$

The integration of Eq. 14.6 results in

$$\int\frac{dp}{\rho}+\frac{V^2}{2}+gy=C \qquad (14.7a)$$

For an **incompressible flow**,

$$\boxed{\frac{p}{\rho}+\frac{V^2}{2}+gy=C} \qquad (14.7b)$$

The constant C in Eqs (14.7a) and (14.7b) has the same value in the entire flow field, since no restriction was made in the choice of dr which was considered as an arbitrary displacement in evaluating the work.

Note: In deriving Eq. (13.8) the displacement **ds** was considered along a streamline. Therefore, the total mechanical energy remains constant everywhere in an inviscid and irrotational flow, while it is constant only along a streamline for an inviscid but rotational flow.

The equation of motion for the flow of an inviscid fluid can be written in a vector form as

$$\frac{D\vec{V}}{Dt} = -\frac{\nabla p}{\rho} + \vec{X}$$

where $\vec{X}$ is the body force vector per unit mass

5.3 Momentum Equation in Integral Form:

Conservation of Momentum: Momentum Theorem

In Newtonian mechanics, the conservation of momentum is defined by Newton's second law of motion.

Newton's Second Law of Motion

- The rate of change of momentum of a body is proportional to the impressed action and takes place in the direction of the impressed action.
- If a force acts on the body ,linear momentum is implied.
- If a torque (moment) acts on the body,angular momentum is implied.

Reynolds Transport Theorem

A study of fluid flow by the Eulerian approach requires a mathematical modeling for a control volume either in differential or in integral form. Therefore the physical statements of the principle of conservation of mass, momentum and energy with reference to a control volume become necessary.

This is done by invoking a theorem known as the Reynolds transport theorem which relates the control volume concept with that of a control mass system in terms of a general property of the system.

Statement of Reynolds Transport Theorem

The theorem states that "the time rate of increase of property N within a control mass system is equal to the time rate of increase of property N within the control volume plus the net rate of efflux of the property N across the control surface".

Equation of Reynolds Transport Theorem

After deriving Reynolds Transport Theorem according to the above statement we get

$$\left(\frac{dN}{dt}\right)_{CMS} = \frac{\partial}{\partial t}\iiint_{CV} \eta\rho \, d\forall + \iint_{CS} \eta\rho \vec{V} \cdot d\vec{A} \qquad (10.9)$$

In this equation

N - flow property which is transported

η - intensive value of the flow property

Application of the Reynolds Transport Theorem to Conservation of Mass and Momentum

Conservation of mass The constancy of mass is inherent in the definition of a control mass system and therefore we can write

$$\left(\frac{dm}{dt}\right)_{CMS} = 0 \qquad (10.13a)$$

To develop the analytical statement for the conservation of mass of a control volume, the Eq. (10.11) is used with N = m (mass) and η = 1 along with the Eq. (10.13a).

This gives

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \, d\forall + \iint_{CS} \rho(\vec{V}_r \cdot d\vec{A}) = 0 \qquad (10.13b)$$

The Eq. (10.13b) is identical to Eq. (10.6) which is the integral form of the continuity equation derived in earlier section. At steady state, the first term on the left hand side of Eq. (10.13b) is zero. Hence, it becomes

$$\iint_{CS} \rho\left(\vec{V}_r \cdot d\vec{A}\right) = 0 \qquad (10.13c)$$

Conservation of Momentum or Momentum Theorem The principle of conservation of momentum as applied to a control volume is usually referred to as the momentum theorem.

Linear momentum The first step in deriving the analytical statement of linear momentum theorem is to write the Eq. (10.11) for the property N as the linear - momentum $m\vec{V}$ and accordingly η as the velocity $(\vec{V})$. Then it becomes

$$\frac{d}{dt}(m\vec{V})_{cm} = \frac{\partial}{\partial t}\iiint_{cv} \vec{V}\rho d\forall + \iint_{cs} \vec{V}\rho(\vec{V}_r.d\vec{A}) \qquad (10.14)$$

The velocity $(\vec{V})$ defining the linear momentum in Eq. (10.14) is described in an inertial frame of reference. Therefore we can substitute the left hand side of Eq. (10.14) by the external forces $\sum \vec{F}$ on the control mass system or on the coinciding control volume by the direct application of Newton's law of motion. This gives

$$\sum \vec{F} = \frac{\partial}{\partial t}\iiint_{cv} \vec{V}\rho d\forall + \iint_{cs} \vec{V}\rho(\vec{V}_r.d\vec{A}) \qquad (10.15)$$

This Equation is the analytical statement of linear momentum theorem.

In the analysis of finite control volumes pertaining to practical problems, it is convenient to describe all fluid velocities in a frame of coordinates attached to the control volume. Therefore, an equivalent form of Eq.(10.14) can be obtained, under the situation, by substituting N as and accordingly η as $\vec{V}_r$, we get

$$\frac{d}{dt}(m\vec{V}_r)_{cm} = \frac{\partial}{\partial t}\iiint_{cv} \vec{V}_r\rho d\forall + \iint_{cs} \vec{V}_r\rho(\vec{V}_r.d\vec{A}) \qquad (10.16)$$

With the help of the Eq. (10.12) the left hand side of Eq. can be written as

$$\frac{d}{dt}(m\vec{V}_r)_{cm} = m\left(\frac{d\vec{V}_r}{dt}\right)_{cm}$$

$$= m\frac{d}{dt}(\vec{V} - \vec{V}_c)_{cm}$$

$$= m\left(\frac{d\vec{V}}{dt}\right)_{cm} - m\vec{a}_c$$

where $a_c\left(= d\vec{V}_c / dt\right)$ is the rectilinear acceleration of the control volume (observed in a fixed coordinate system) which may or may not be a function of time. From Newton's law of motion

$$m\left(\frac{d\vec{V}}{dt}\right)_{CMS} = \sum \vec{F}$$

Therefore, $$m\left(\frac{d\vec{V}_r}{dt}\right)_{CMS} = \sum \vec{F} - m\vec{a}_o \qquad (10.17)$$

The Eq. (10.16) can be written in consideration of Eq. (10.17) as

$$\sum \vec{F} - m\vec{a}_o = \frac{\partial}{\partial t}\iiint_{CV} \vec{V}_r \rho d\forall + \iint_{CS} \vec{V}_r \rho(\vec{V}_r . d\vec{A}) \qquad (10.18a)$$

At steady state, it becomes

$$\sum \vec{F} - m\vec{a}_o = \iint_{CS} \vec{V}_r \rho(\vec{V}_r . d\vec{A}) \qquad (10.18b)$$

In case of an inertial control volume (which is either fixed or moving with a constant rectilinear velocity), $\vec{a}_o = 0$ and hence Eqs (10.18a) and (10.18b) becomes respectively

$$\sum \vec{F} = \frac{\partial}{\partial t}\iiint_{CV} \vec{V}_r \rho d\forall + \iint_{CS} \vec{V}_r \rho(\vec{V}_r . d\vec{A}) \qquad (10.18c)$$

and $$\sum \vec{F} = \iint_{CS} \vec{V}_r \rho(\vec{V}_r . d\vec{A}) \qquad (10.18d)$$

The Eqs (10.18c) and (10.18d) are the useful forms of the linear momentum theorem as applied to an inertial control volume at unsteady and steady state respectively, while the Eqs (10.18a) and (10.18b) are the same for a non-inertial control volume having an arbitrary rectilinear acceleration.

In general, the external forces $\sum \vec{F}$ in Eqs (10.14, 10.18a to 10.18c) have two components - the body force and the surface force. Therefore we can write

$$\sum \vec{F} = \iiint_{CV} \vec{F}_B d\forall + \vec{F}_S \qquad (10.18e)$$

where $\vec{F}_B$ is the body force per unit volume and $\vec{F}_S$ is the area weighted surface force.

5.4 Angular Momentum Equation in Integral Form:

Angular Momentum

The angular momentum or moment of momentum theorem is also derived from Eq.(10.10) in consideration of the property N as the angular momentum and accordingly η as the angular momentum per unit mass. Thus,

$$\frac{d}{dt}(A_{CMS}) = \frac{\partial}{\partial t}\iiint_{CV} \rho(\vec{r} \times \vec{V}_r)d\forall + \iint_{CS} (\vec{r} \times \vec{V}_r)\rho(\vec{V}_r.d\vec{A}) \qquad (10.19)$$

where $A_{\text{Control mass system}}$ is the **angular momentum of the control mass system**. . It has to be noted that the origin for the angular momentum is the origin of the position vector $\vec{r}$

The term on the left hand side of Eq.(10.19) is the time rate of change of angular momentum of a control mass system, while the first and second terms on the right hand side of the equation are the time rate of increase of angular momentum within a control volume and rate of net efflux of angular momentum across the control surface.

The velocity $(\vec{V})$ defining the angular momentum in Eq.(10.19) is described in an inertial frame of reference.Therefore, the term $d/dt(A_{CMS})$ can be substituted by the net moment ΣM applied to the system or to the coinciding control volume. Hence one can write Eq. (10.19) as

$$\sum M = \frac{\partial}{\partial t}\iiint_{CV} \rho(\vec{r} \times \vec{V})d\forall + \iint_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r.d\vec{A}) \qquad (10.20a)$$

At steady state

$$\frac{\partial}{\partial t}\iiint_{CV} \rho(\vec{r} \times \vec{V})d\forall = 0$$

$$\sum M = \iint_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r.d\vec{A}) \qquad (10.20b)$$

Chapter 6
Inviscid Incompressible Flow

6.1 Condition on Velocity for Incompressible Flow:

Analysis of Inviscid, Incompressible, Irrotation Flows

Incompressible flow is a constant density flow.

Let us visualize a fluid element of defined mass, moving along a streamline in an incompressible flow.

Due to constant density , we can write

$$\nabla \cdot \vec{V} = 0 \qquad (20.1)$$

Irrotational Flow

- **if the fluid element does not rotate as it moves along the streamline, or to be precise, if its motion is translational (and deformation with no rotation) only, the flow is termed as irrotational.**

The **rate of rotation of the fluid element** can be measured as the **average rate of rotation of two perpendicular line segments.**

The average rate of rotation ω_z about z-axis is expressed in terms of the gradients of velocity components as

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

Similarly, the other two components of rotation are

$$\omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right] \quad \text{and} \quad \omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]$$

ω_x , ω_y and ω_z are components of $\vec{\omega}$

$$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{V})$$

In a two-dimensional flow, ω_z is the only non-trivial component of the rate of rotation called in-plane component of vorticity and computed as $\frac{1}{2}(\nabla \times \vec{V})\cdot\hat{k}$

Thus **for irrotational flow, vorticity is zero i.e.** $\vec{\omega} = 0$

6.2 Laplace Equation:

Potential Flow Theory

Let us imagine a pathline of a fluid particle shown in Fig. 20.1.

Rate of spin of the particle is ω_z . The flow in which this spin is zero throughout is known as irrotationalflow.
For irrotational flows, $\nabla \times \vec{V} = 0$

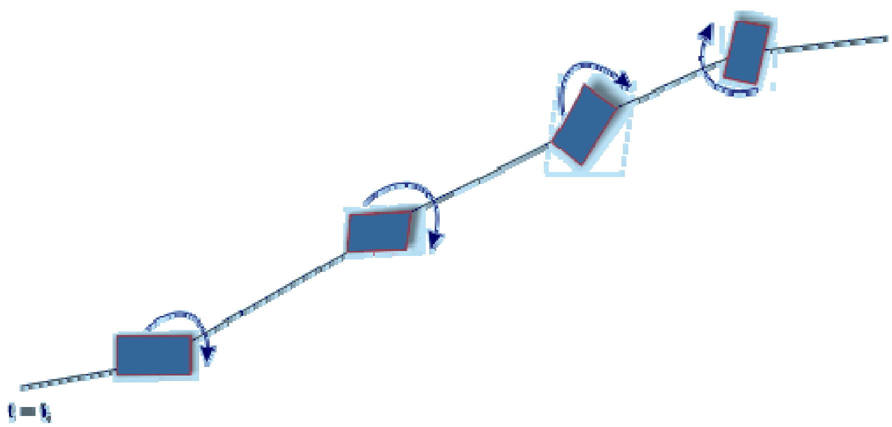


Fig 20.1 Pathline of a Fluid Particle

Velocity Potential and Stream Function

Since for irrotational flows $\nabla \times \vec{V} = 0$.

the velocity for an irrotational flow, can be expressed as the gradient of a scalar function called the velocity potential, denoted by Φ

$$\vec{V} = \nabla \phi \qquad (20.2)$$

Combination of Eqs (20.1) and (20.2) yields

$$\nabla^2 \phi = 0 \qquad \text{(Laplace's equation)} \qquad (20.3)$$

For irrotational flows

$$\bar{\omega}=0$$

For two-dimensional case (as shown in Fig 20.1)

$$\omega=\frac{1}{2}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]=0$$

$$\Rightarrow\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$$

$$-\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)=C \qquad \left[u=\frac{\partial \psi}{\partial y};v=-\frac{\partial \psi}{\partial x}\right]$$

$$-\left(+\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial x^2}\right)=0$$

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=0$$

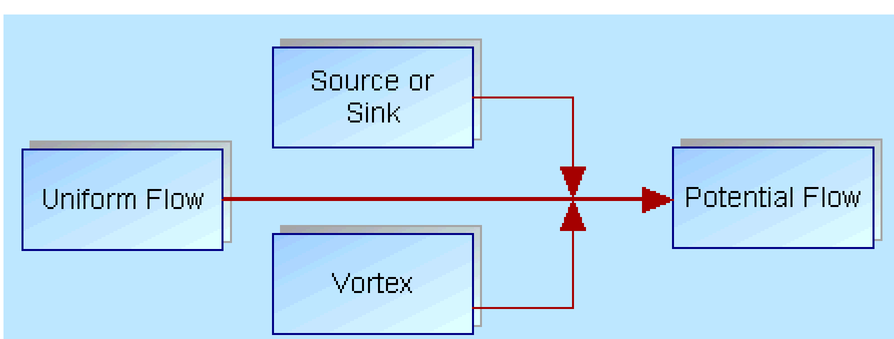
$$\nabla^2\psi=0$$

which is again Laplace's equation.

- From Eq. (20.3) we see that **an inviscid, incompressible, irrotational flow is governed by Laplace's equation.**
- Laplace's equation is linear, hence any number of particular solutions of Eq.(20.3) added together will yield another solution .
- A complicated flow pattern for an inviscid, incompressible, irrotational flow can be synthesized by adding together a number of elementary flows (provided they are also inviscid, incompressible and irrotational)----- **The Superposition Principle**

The analysis of Laplace's Eq. (20.3) and finding out the potential functions are known as Potential Flow Theory and the inviscid, incompressible, irrotational flow is often called as Potential Flow .

There are some elementary flows which constitute several complex potential-flow problems.



6.3 Potential Function:

Refer to Section 6.2

6.4 Stream Function:

Stream Function

Let us consider a two-dimensional incompressible flow parallel to the x - y plane in a rectangular cartesian coordinate system. The flow field in this case is defined by

$u = u(x, y, t)$
$v = v(x, y, t)$
$w = 0$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (10.1) \qquad\qquad (10.1)$$

If a function $\psi(x, y, t)$ is defined in the manner

$$u = \frac{\partial \psi}{\partial y} \qquad (10.2a)$$

$$v = -\frac{\partial \psi}{\partial x} \qquad (10.2b)$$

so that it automatically satisfies the equation of continuity (Eq. (10.1)), then the function is known as stream function.
Note that for a **steady flow, ψ is a function of two variables x and y only.**

Constancy of ψ on a Streamline

Since ψ is a point function, it has a value at every point in the flow field. Thus a change in the stream function ψ can be written as

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy$$

The equation of a streamline is given by

$$\frac{u}{dx} = \frac{v}{dy} \qquad \text{or} \qquad udy - vdx = 0 \text{(since tangent dy/dx equals the velocity v/u)}$$

It follows that dψ = 0 on a streamline.This implies the value of ψ is constant along a streamline. Therefore, the equation of a streamline can be expressed in terms of stream function as

$$\psi(x, y) = \text{constant} \qquad (10.3)$$

Once the function ψ is known, streamline can be drawn by joining the same values of ψ in the flow field.

Stream function for an irrotational flow

In case of a two-dimensional irrotational flow

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

$$\Rightarrow -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\Rightarrow \psi_{xx} + \psi_{yy} = 0$$

$$\Rightarrow \nabla^2 \psi = 0$$

Conclusion drawn: For an irrotational flow, stream function satisfies the Laplace's equation

Physical Significance of Stream Funtion ψ

Figure 10.1 illustrates a two dimensional flow.

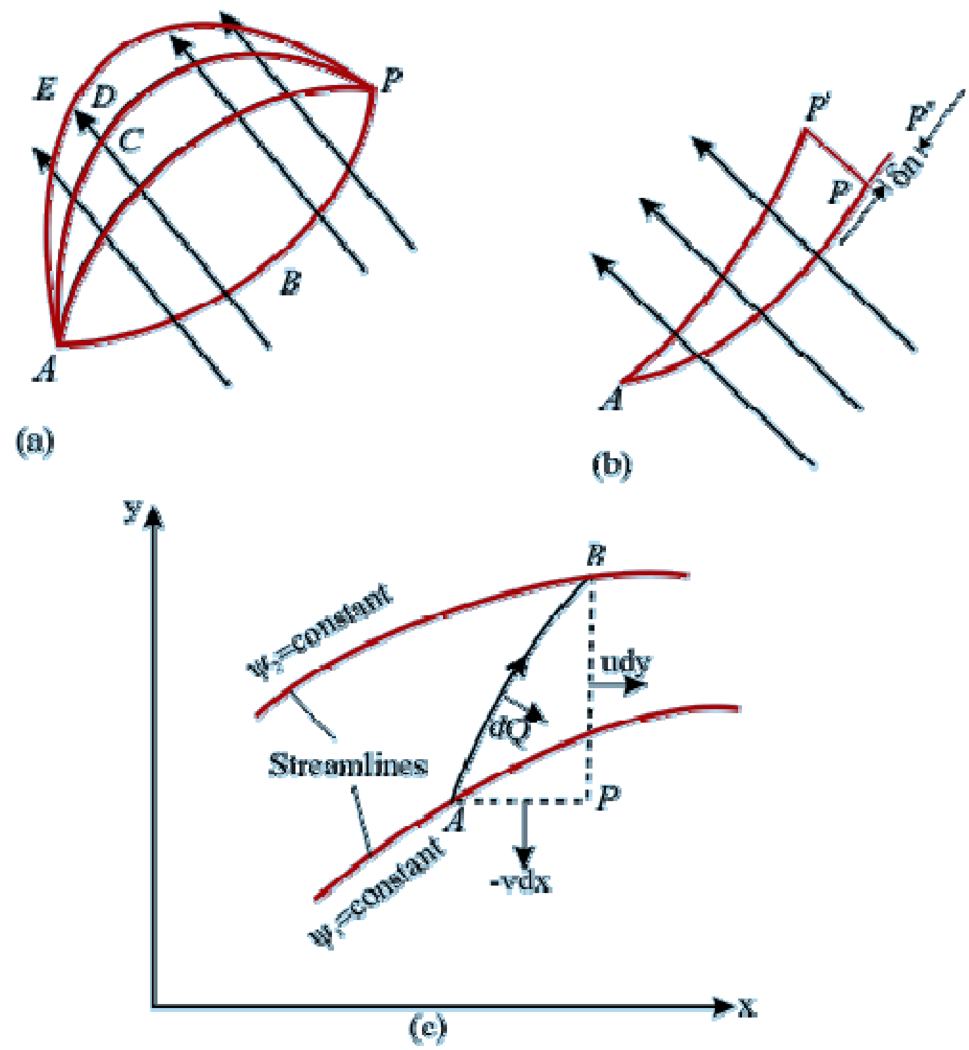


Fig 10.1 Physical Interpretation of Stream Function

Let A be a fixed point, whereas P be any point in the plane of the flow. The points A and P are joined by the arbitrary lines ABP and ACP. For an incompressible steady flow, the volume flow rate across ABP into the space ABPCA (considering a unit width in a direction perpendicular to the plane of the flow) must be equal to that across ACP. A number of different paths connecting A and P (ADP, AEP,...) may be imagined but the volume flow rate across all the paths would be the same. This implies that the **rate of flow across any curve between A and P depends only on the end points A and P.**

Since A is fixed, the rate of flow across ABP, ACP, ADP, AEP (any path connecting A and P) is a function only of the position P. This function is known as the **stream function ψ.**

The value of ψ at P represents the volume flow rate across any line joining P to A.
The value of ψ at A is made arbitrarily zero. If a point P' is considered (Fig. 10.1b),PP' being along a streamline, then the rate of flow across the curve joining A to P' must be the same as across AP, since, by the definition of a streamline, there is no flow across PP'

The value of ψ thus remains same at P' and P. Since P' was taken as any point on the streamline through P, it follows that ψ is constant along a streamline. Thus the flow may be represented by a series of streamlines at equal increments of ψ.

In fig (10.1c) moving from A to B net flow going past the curve AB is

$$\int dQ = \int_A^B (u\,dy - v\,dx)$$

$$= \int_A^B (\psi_y\,dy + \psi_x\,dx) \qquad \left[\text{since } u = \psi_y \text{ and } v = -\psi_x\right]$$

$$\int dQ = \int_A^B d\psi$$

$$\therefore Q = \int_A^B d\psi = \psi_2 - \psi_1$$

The stream function, in a polar coordinate system is defined as

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad V_\theta = -\frac{\partial \psi}{\partial r}$$

The expressions for V_r and V_θ in terms of the stream function automatically satisfy the equation of continuity
given by

$$\frac{\partial}{\partial r}(V_r r) + \frac{\partial}{\partial \theta}(V_\theta) = 0$$

Stream Function in Three Dimensional and Compressible Flow

Stream Function in Three Dimensional Flow

In case of a three dimensional flow, it is not possible to draw a streamline with a single stream function.

An axially symmetric three dimensional flow is similar to the two-dimensional case in a sense that the flow field is the same in every plane containing the axis of symmetry.

The equation of continuity in the cylindrical polar coordinate system for an incompressible flow is given by the following equation

$$\frac{\partial V_r}{\partial r}+\frac{V_r}{r}+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{\partial V_z}{\partial z}=0$$

For an axially symmetric flow (the axis r = 0 being the axis of symmetry), the term $\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}=0$,and simplified equation is satisfied by functions defined as

$$rV_r=-\frac{\partial \psi}{\partial x},\qquad rV_x=\frac{\partial \psi}{\partial r} \tag{10.4}$$

The function ψ , defined by the Eq.(10.4) in case of a three dimensional flow with an axial symmetry, is called the **stokesstream function.**

Stream Function in Compressible Flow

For compressible flow, stream function is related to mass flow rate instead of volume flow rate because of the extra density term in the continuity equation (unlike incompressible flow)

The continuity equation for a steady two-dimensional compressible flow is given by

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0$$

Hence a stream function ψ is defined which will satisfy the above equation of continuity as

$$\rho u=\rho_0\frac{\partial \psi}{\partial y}$$
$$\rho v=-\rho_0\frac{\partial \psi}{\partial x} \tag{10.5}$$
[where ρ_0 is a reference density]

ρ_0 is used to retain the unit of ψ same as that in the case of an incompressible flow. Physically, the difference in stream function between any two streamlines multiplied by the reference density ρ_0 will give the mass flow rate through the passage of unit width formed by the streamlines.

6.5 Basic Elementary Flows:

Uniform Flow

- Velocity does not change with y-coordinate
- There exists only one component of velocity which is in the x direction.
- Magnitude of the velocity is U_0 .

Since $\vec{V} = \nabla\phi$

$$\hat{i}U_0 + \hat{j}0 = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y}$$

or,

$$\frac{\partial\phi}{\partial x} = U_0, \quad \frac{\partial\phi}{\partial y} = 0$$

Thus,

$$\phi = U_0 x + C_1 \qquad (20.4)$$

Using stream function ψ for uniform flow

$$\frac{\partial\psi}{\partial x} = 0, \quad \frac{\partial\psi}{\partial y} = U_0$$

so

$$\psi = U_0 y + K_1 \qquad (20.5)$$

The constants of integration C_1 and K_1 are arbitrary.

The values of ψ and Φ for different streamlines and velocity potential lines may change but flow pattern is unaltered

. The **constants of integration may be omitted, without any loss of generality** and it is possible to write

$$\psi = U_0 y, \quad \phi = U_0 x \qquad (20.6)$$

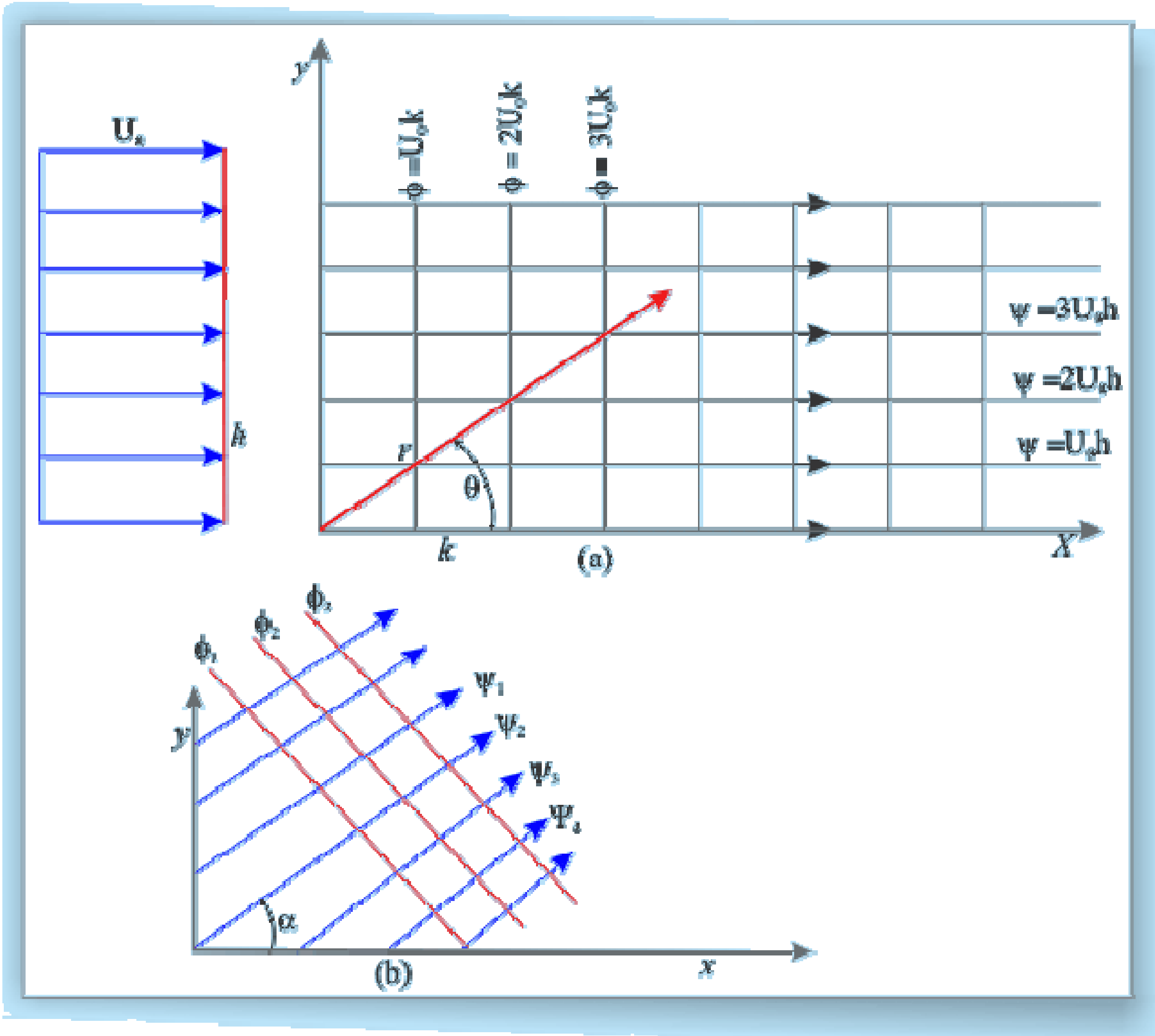


Fig 20..2 (a) Flownet for a Uniform Stream (b) Flownet for uniform Stream with an Anglea with x-axis

These are plotted in Fig. 20.2(a) and consist of a rectangular mesh of straight streamlines and orthogonal straight potential-lines (remember **streamlines and potential lines are always orthogonal**). It is conventional to put arrows on the streamlines showing the direction of flow.

In terms of polar (r - θ) coordinate, Eq. (20.6) becomes

$$\psi = U_0 r \sin\theta, \quad \phi = U_0 r\cos\theta \qquad (20.7)$$

Flow at an angle

If we consider a uniform stream at an angle α to the x-axis as shown in Fig. 20.2b. we require that

$$u = U_0 \cos\alpha = \frac{d\psi}{dy} = \frac{d\phi}{dx}$$

and

$$v = U_0 \sin\alpha = -\frac{d\psi}{dx} = \frac{d\phi}{dy} \qquad (20.8)$$

Integrating. we obtain for a uniform velocity U_0 at an angle α, the stream function and velocity potential respectively as

$$\boxed{\psi = U_0(y\cos\alpha - x\sin\alpha), \quad \phi = U_0(x\cos\alpha + y\sin\alpha)} \qquad (20.9)$$

Source or Sink

Source flow -

- A flow with straight streamlines emerging from a point.
- Velocity along each streamline varies inversely with distance from the point (shown in Fig. 20.3).
- Only the radial component of velocity is non-trivial. (v_θ=0, v_z=0).

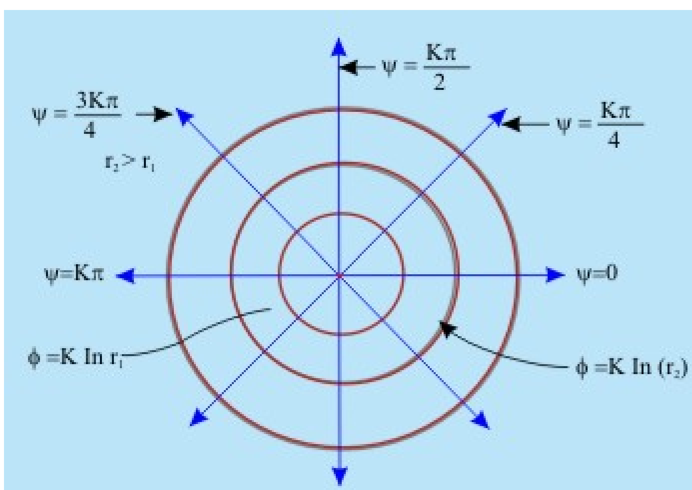


Fig 20.3 Flownet for a source flow

In a steady source flow the amount of fluid crossing any given cylindrical surface of radius r and unit length is constant ($\dot{m}$)

that is ($\dot{m}$)$_{in}$= ($\dot{m}$)$_{out}$

$$\dot{m} = 2\pi r v_r \rho$$

$$v_r = \frac{\dot{m}}{2\pi\rho}\cdot\frac{1}{r} = \frac{\Lambda}{2\pi}\cdot\frac{1}{r} = \frac{K}{r} \qquad (20.10a)$$

(which shows that velocity is inversely proportional to the distance)

where, K is the source strength $K = \frac{\dot{m}}{2\pi\rho} = \frac{\Lambda}{2\pi}$ and Λ is the volume flow rate

The **definition of stream function in cylindrical polar coordinate** states that

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \qquad (20.11)$$

For the source flow,

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{K}{r} \qquad (20.12)$$

$$-\frac{\partial \psi}{\partial r} = 0 \qquad (20.13)$$

Combining Eqs (20 .12) and (20.13) , we get

$$\psi = K\theta + C_1 \qquad (20.14)$$

Thus

$$\text{if} \quad \psi = k\tan^{-1}\left(\frac{y}{x}\right)$$

$$u = \frac{\partial \psi}{\partial y} = k\left(\frac{x}{x^2+y^2}\right) \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} = k\left(\frac{y}{x^2+y^2}\right)$$

Because the flow is irrotational, we can write

$$r v_r + \theta v_\theta = r\frac{\partial \phi}{\partial r} + \theta\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

or

$$v_r = \frac{\partial \phi}{\partial r} \quad \text{and} \quad v_\theta = 0 = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

or

$$v_r = \frac{\partial \phi}{\partial r} = \frac{K}{r} \quad \text{or} \quad \phi = K\ln r + C_2 \qquad (20.15)$$

The integration constants C_1 and C_2 in Eqs (20.14) and (20.15) have no effect on the basic structure of velocity and pressure in the flow.

The equations for streamlines and velocity potential lines for source flow become

$$\phi = K \ln r \qquad \psi = K\theta \tag{20.16}$$

K = source strength and is proportional to Λ
Λ = the rate of volume flow from the source per unit depth perpendicular to the page

Sink flow

- When Λ **is negative** , we get sink flow,
- here the flow is in the opposite direction of the source flow.

In Fig. 20.3, the point 0 is the origin of the radial streamlines. We visualize that point O is a point source or sink that induces radial flow in the neighbourhood .

The point source or sink is a point of singularity in the flow field (because v_r becomes infinite).

The stream **function and velocity potential function are**

$$\phi = -K \ln r \qquad \psi = -K\theta \tag{20.17}$$

Concept of Circulation in a Free Vortex Flow

Free Vortex Flow

- Fluid particles move in circles about a point.
- The only non-trivial velocity component is tangential.
- This tangential speed varies with radius r so that same circulation is maintained.
- Thus,all the **streamlines are concentric circles** about a given point where the velocity along each streamline is inversely proportional to the distance from the centre. This **flow is necessarily irrotational.**

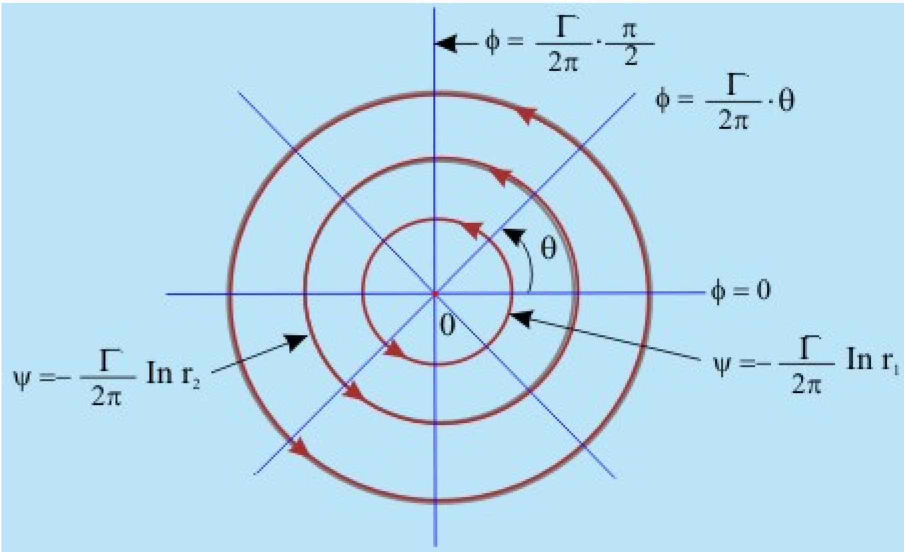


Fig 21.1 Flownet for a vortex (free vortex)

Velocity components

In a purely circulatory (free vortex flow) motion, the tangential velocity can be written as

$$v_\theta = \frac{\text{Circulation constant}}{r}$$

or,

$$v_\theta = \frac{\Gamma/2\pi}{r} \quad \text{where } \Gamma \text{ is circulation} \qquad (21.1)$$

For purely circulatory motion we can also write

$$v_r = 0 \qquad (21.2)$$

Stream Function

Using the definition of stream function, we can write

$$v_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{and} \qquad v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

Combining Eqs (21.1) and (21.2) with the above said relations for stream function, it is possible to write

$$\psi = -\frac{\Gamma}{2\pi}\ln r + C_1 \tag{21.3}$$

Velocity Potential Function

Because of irrotationality, it should satisfy

$$\hat{r}v_r + \hat{\theta}v_\theta = \hat{r}\frac{\partial \phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Eqs (21.1) and (21.2) and the above solution of Laplace's equation yields

$$\phi = \frac{\Gamma}{2\pi}\theta + C_2 \tag{21.4}$$

Since, the integration constants C_1 and C_2 have no effect on the structure of velocities or pressures in the flow. We can ignore the integration constants without any loss of generality.

It is clear that the **streamlines for vortex flow are circles while the potential lines are radial** . **These are given by**

$$\boxed{\psi = \frac{\Gamma}{2\pi}\ln r \qquad \text{and} \qquad \phi = \frac{\Gamma}{2\pi}\theta} \tag{21.5}$$

- In Fig. 21.1, point 0 can be imagined as a point vortex that induces the circulatory flow around it.
- The point vortex is a singularity in the flow field (v_θ becomes infinite).
- Point 0 is simply a point formed by the intersection of the plane of a paper and a line perpendicular to the plane.
- This line is called vortex filament of strength Γ where Γ is the circulation around the vortex filament .

Circulation is defined as

$$\Gamma = \int \vec{V} \cdot d\vec{s} \tag{21.6}$$

This circulation constant denotes the algebraic strength of the vortex filament contained within the closed curve. From Eq. (21.6) we can write

$$\Gamma = \int \vec{V} \cdot d\vec{s} = \int (u\,dx + v\,dy + w\,dz)$$

For a two-dimensional flow

$$\Gamma = \int (u\,dx + v\,dy)$$

or,

$$\Gamma = \int V \cos\alpha\, ds \quad \text{(according to Fig. 21.2)} \qquad (21.7)$$

Consider a fluid element as shown in Fig. 21.2. **Circulation is positive in the anticlockwise direction** (not a mandatory but general convention).

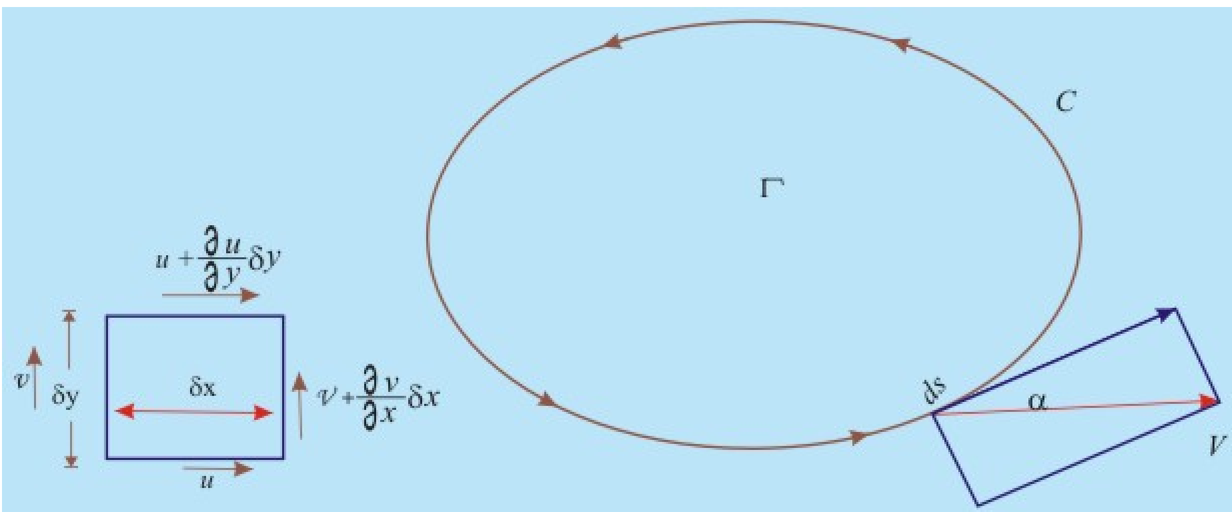


Fig 21.2 Circulation in a flow field

$$\delta\Gamma = u\delta x + \left(v + \frac{dv}{dx}\delta x\right)\delta y - \left(u + \frac{du}{dy}\delta y\right)\delta x - v\delta y$$

$$= \delta x \delta y \left(\frac{dv}{dx} - \frac{du}{dy}\right) = \delta x \delta y (2\omega_z) \qquad \text{where } \delta A = \delta x \delta y$$

After simplification

$$\frac{\delta\Gamma}{\delta A} = 2\omega_z = \Omega_z \qquad (21.8)$$

Physically, **circulation per unit area is the vorticity of the flow** .

Now, for a free vortex flow, the tangential velocity is given by Eq. (21.1) as

$$v_\theta = \frac{\Gamma/2\pi}{r} = \frac{C}{r}$$

For a circular path (refer Fig.21.2)

$$v_\theta = \frac{\Gamma/2\pi}{r} = \frac{C}{r}$$

Thus,

$$\Gamma = \int_0^{2\pi} v r d\theta$$
$$= 2\pi C$$

Therefore

$$G = 2\pi C \qquad (21.9)$$

It may be noted that although free vortex is basically an irrotational motion, the circulation for a given path containing a singular point (including the origin) is constant (2πC) and independent of the radius of a circular streamline.

- However, **circulation calculated in a free vortex flow along any closed contour excluding the singular point (the origin), should be zero.**

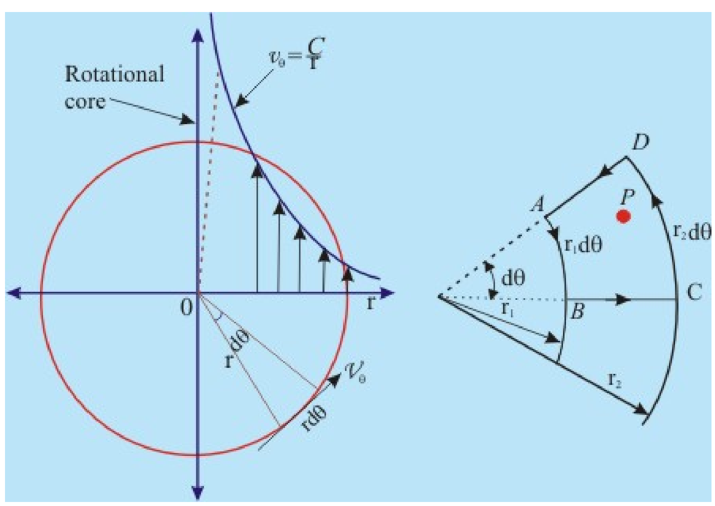


Fig 21.3 (a) Free Vortex Flow

Considering Fig 21.3 (a) and taking a closed contour ABCD in order to obtain circulation about the point, P around ABCD it may be shown that

$$\Gamma_{ABCD} = \Gamma_{AB} + \Gamma_{BC} + \Gamma_{CD} + \Gamma_{DA} = -\frac{C}{r_1} r_1 d\theta + 0 + \frac{C}{r_2} r_2 d\theta + 0 = 0$$

Forced Vortex Flow

- If there exists a solid body rotation at constant ω (induced by some external mechanism), the flow should be called a forced vortex motion (Fig. 21.3 (b).

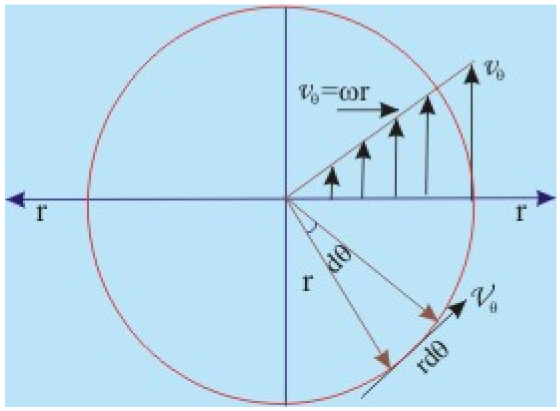


Fig 21.3 (b) Forced Vortex Flow

we can write

$v_\theta = \omega r$ and

$$\Gamma = \int v_t ds = \int_0^{2\pi} \omega r . r d\theta = 2\pi r^2 \omega \qquad (21.10)$$

Equation (21.10) predicts that

1. The circulation is zero at the origin
2. It increases with increasing radius.
3. The variation is parabolic.

It may be mentioned that the **free vortex (irrotational) flow at the origin is impossible** because of mathematical singularity. However, physically there should exist a rotational (forced vortex) core which is shown by the dotted line (in Fig. 21.3a).

Below are given two statements which are related to Kelvin's circulation theorem (stated in 1869) and Cauchy's theorem on irrotational motion (stated in 1815) respectively

1. The circulation around any closed contour is invariant with time in an inviscid fluid.--- **Kelvin's Theorem**
2. A body of inviscid fluid in irrotational motion continues to move irrotationally.------------ **Cauchy's Theorem**

6.6 Superimposition of Elementary Flows:

Combination of Fundamental Flows

1) Doublet

We can now form different flow patterns by superimposing the velocity potential and stream functions of the elementary flows stated above.

In order to develop a doublet, imagine a source and a sink of equal strength K at equal distance s from the origin along x-axis as shown in Fig. 21.4.

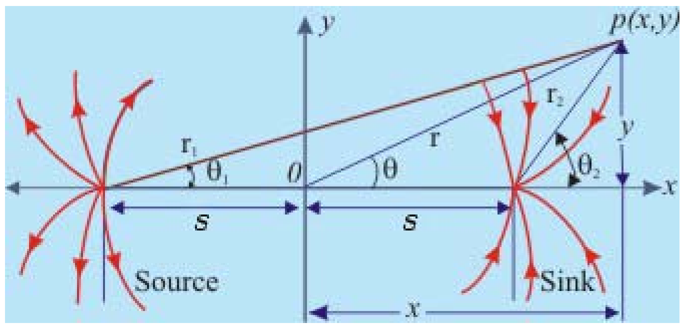


Fig 21.4 Superposition of a Source and a Sink

From any point p(x, y) in the field, r_1 and r_2 are drawn to the source and the sink. The polar coordinates of this point (r, θ) have been shown.

The potential functions of the two flows may be superimposed to describe the potential for the combined flow at P as

$$\phi = k \ln r_1 - k \ln r_2 \tag{21.11}$$

Similarly,

$$\psi = k(\theta_1 - \theta_2) = -k\alpha \tag{21.12}$$

where $\alpha = (\theta_2 - \theta_1)$

Expanding θ_1 and θ_2 in terms of coordinates of p and s

$$\tan\theta_1 = \frac{y}{x+s} \qquad \tan\theta_2 = \frac{y}{x-s} \tag{21.13}$$

$$r_1 = \sqrt{\left(r^2 + s^2 + 2rs\cos\theta\right)} \quad \text{and} \quad r_2 = \sqrt{\left(r^2 + s^2 - 2rs\cos\theta\right)} \tag{21.14}$$

Using

$$\tan(\theta_2-\theta_1)=\frac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_1\tan\theta_2}$$

we find

$$\tan\alpha=\left[\frac{yx+ys-yx+ys}{x^2-s^2}\right]\Big/\left(1+\frac{y^2}{x^2-s^2}\right)$$

or, $$\tan\alpha=\frac{2ys}{x^2+y^2-s^2}$$

Hence the stream function and the velocity potential function are formed by combining Eqs (21.12) and (21.13), as well as Eqs(21.11) and (21.14) respectively

$$\psi=-k\alpha$$

Hence $$\psi=-k\tan^{-1}\left(\frac{2ys}{x^2+y^2-s^2}\right)$$ ------- **Stream Function** (21.15)

$$\phi=k\ln r_1-k\ln r_2=k\ln\frac{r_1}{r_2}$$

$$\phi=\frac{k}{2}\ln\left(\frac{r^2+s^2+2rs\cos\theta}{r^2+s^2-2rs\cos\theta}\right)$$ ----- **Potential Function** (21.16)

Doublet is a special case when a source as well as a sink are brought together in such a way that

- $s\rightarrow 0$ and at the same time the
- strength $\frac{\Lambda}{\pi}(k/2)$ is increased to an infinite value.

These are assumed to be accomplished in a manner which makes the product of s and $\frac{\Lambda}{\pi}(k/2)$ (in limiting case) a finite value c

This gives us

$$\psi=-\frac{\chi\sin\theta}{r}\qquad\text{and}\qquad\phi=\frac{\chi\cos\theta}{r}$$

Streamlines, Velocity Potential for a Doublet

We have seen in the last lecture that the streamlines associated with the doublet are

$$-\frac{Z\sin\theta}{r} = C_1$$

If we replace sinθ by y/r, and the minus sign be absorbed in C_1 , we get

$$-\frac{Z\sin\theta r}{r^2} = C_1$$

$$\Rightarrow \frac{Zy}{r^2} = C_1 \qquad (21.17a)$$

Putting $r^2 = x^2 + y^2$ we get

$$x^2 + y^2 - \frac{Z}{C_1}y = 0 \qquad (21.17b)$$

Equation (21.17b) represents a family of circles with

- **radius :** $\frac{Z}{2C_1}$
- **centre :** $\left(0, \frac{Z}{2C_1}\right)$

- For x = 0, there are two values of y, one of them=0.
- The centres of the circles fall on the y-axis.
- On the circle, where y = 0, x has to be zero for all the values of the constant.
- family of circles formed(due to different values of C_1) is tangent to x-axis at the origin.

These streamlines are illustrated in Fig. 21.5.

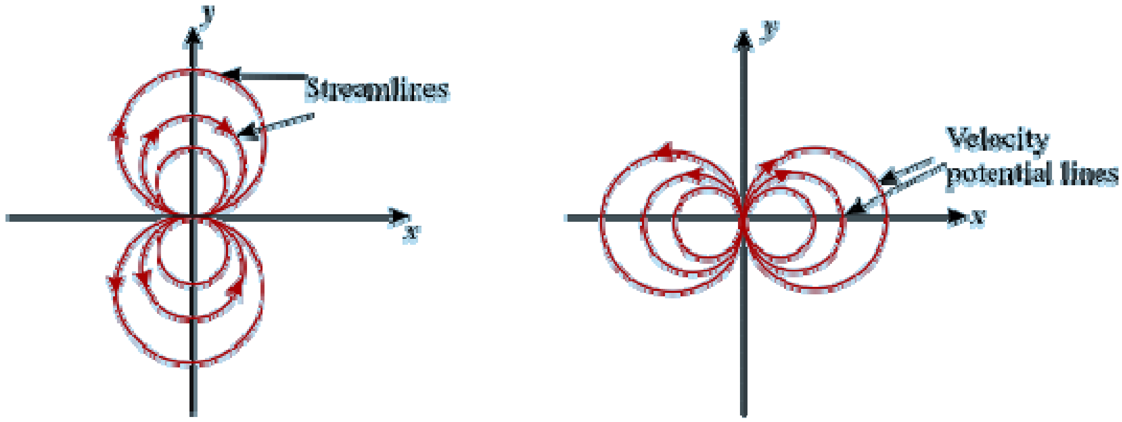


Fig 21.5 Streamlines and Velocity Potential Lines for a Doublet

Due to the initial positions of the source and the sink **in the development of the doublet** , it is certain that

- **the flow will emerge in the negative x direction from the origin**

and

- **it will converge via the positive x direction of the origin.**

Velocity potential lines

$$\frac{\chi \cos\theta}{r} = K_1$$

In cartresian coordinate the equation becomes

$$x^2 + y^2 - \frac{\chi}{K_1}x = 0 \qquad (21.18)$$

Once again we shall obtain **a family of circles**

- **radius:** $\frac{\chi}{2K_1}$
- **centre:** $\left(\frac{\chi}{2K_1}, 0\right)$
- The centres will fall on x-axis.
- For y = 0 there are two values of x, one of which is zero.
- When x = 0, y has to be zero for all values of the constant.
- These circles are tangent to y-axis at the origin.

In addition to the determination of the stream function and velocity potential, it is observed that for a doublet

$$v_r = \frac{d\phi}{dr} = -\frac{\chi \cos\theta}{r^2}$$

As the centre of the doublet is approached; the radial velocity tends to be infinite.

It shows that the doublet flow has a singularity.

Since the circulation about a singular point of a source or a sink is zero for any strength, it is obvious that the **circulation about the singular point in a doublet flow must be zero i.e. doublet flow Γ=0**

$$\Gamma = \int \vec{V} . d\vec{s} = 0 \qquad (21.19)$$

Applying Stokes Theorem between the line integral and the area-integral

$$\Gamma = \iint \left(\nabla \times \vec{V}\right) d\vec{A} = 0 \qquad (21.20)$$

From Eq. 21.20 the obvious conclusion is $\nabla \times \vec{V} = 0$ i.e., **doublet flow is an irrotational flow.**

Flow About a Cylinder without Circulation

- Inviscid-incompressible flow about a cylinder in uniform flow is equivalent to the **superposition of a uniform flow and a doublet.**
- The doublet has its axis of development parallel to the direction of the uniform flow (x-axis in this case).
- The potential and stream function for this flow will be the sum of those for uniform flow and doublet.

Potential Function

$$\phi = U_0 x + \frac{\chi \cos\theta}{r}$$

Stream function

$$\psi = U_0 y - \frac{\chi \sin\theta}{r}$$

Streamlines

In two dimensional flow, a streamline may be interpreted as

- the edge of a surface, on which the velocity vector is always tangential.

and

- there is no flow in the direction normal to the surface (characteristic of a solid impervious boundary).

Hence, a **streamline** may also be considered as the **contour of an impervious two-dimensional body** .

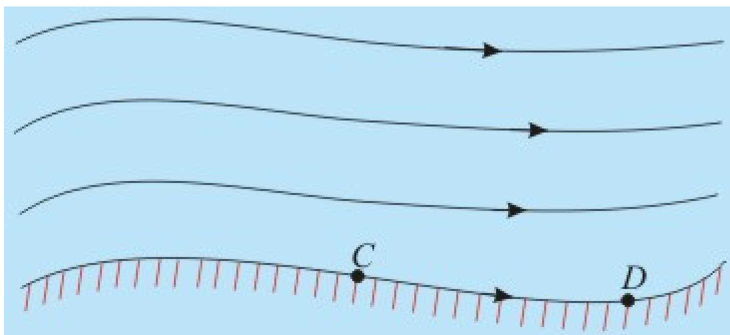


Fig 22.1 Surface Streamline

Figure 22.1 shows a set of streamlines.

1. The **streamline C-D** may be considered as the **edge of a two-dimensional body .**
2. other streamlines form the flow about the boundary.

In order to form a flow about the body of interest, a streamline has to be determined which encloses an area whose shape is of practical importance in fluid flow. This streamline describes the boundary of a two-dimensional solid body. The remaining streamlines outside this solid region, constitute the flow about this body.

If we look for the streamline whose value is zero, we will obtain

$$U_0 y - \frac{z \sin\theta}{r} = 0 \qquad (22.1)$$

replacing y by rsinθ, we have

$$\sin\theta\left(U_0 r - \frac{z}{r}\right) = 0 \qquad (22.2)$$

Solution of Eq. 22.2

1. If θ = 0 or θ = π, the equation is satisfied. This indicates that the x-axis is a part of the streamline Ψ = 0.
2. When the quantity in the parentheses is zero, the equation is **identically satisfied** . Hence it follows that

$$r = \left(\frac{\chi}{U_0}\right)^{1/2} \qquad (22.3)$$

Interpretation of the solution

There is a circle of radius $r = \left(\frac{\chi}{U_0}\right)^{1/2}$ which is an intrinsic part of the streamline Ψ = 0.

This is shown in Fig.22.2

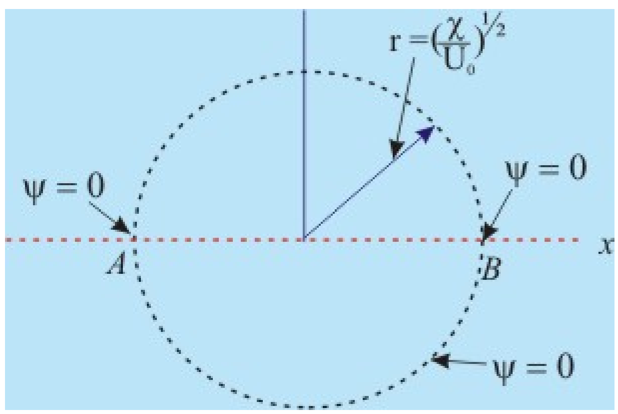


Fig 22.2 Streamline ψ = 0 in a Superimposed Flow of Doublet and Uniform Stream

Stagnation Points

Let us look at the points of intersection of the circle and x- axis , i.e. the points A and B in the above figure. The polar coordinate of these points are

$r - \left(\frac{\chi}{U_0}\right)^{1/2}, \theta - \pi$ for point A

$r = \left(\frac{\chi}{U_0}\right)^{1/2}, \theta = 0$ for point B

The velocity at these points are found out by taking partial derivatives of the velocity potential in two orthogonal directions and then substituting the proper values of the coordinates.

Since, $\phi = U_0 r\cos\theta + \frac{\Gamma\cos\theta}{r}$

$$v_r = \frac{\partial\phi}{\partial r} = U_0\cos\theta - \frac{\Gamma\cos\theta}{r^2} \quad (22.4a)$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

$$= \frac{1}{r}\left[-U_0 r\sin\theta - \frac{\Gamma\sin\theta}{r}\right] \quad (22.4b)$$

$$= -U_0\sin\theta - \frac{\Gamma\sin\theta}{r^2}$$

At point A $\left[\theta = \pi, r = \left(\frac{\Gamma}{U_0}\right)^{1/2}\right]$

$$v_r = 0, v_\theta = 0$$

At point B $\left[\theta = 0, r = \left(\frac{\Gamma}{U_0}\right)^{1/2}\right]$

$$v_r = 0, v_\theta = 0$$

The points A and B are the stagnation points through which the flow divides and subsequently reunites forming a zone of circular bluff body.

The circular region, enclosed by part of the streamline $\psi = 0$ could be imagined as a solid cylinder in an inviscid flow. At a large distance from the cylinder the flow is moving uniformly in a cross-flow configuration.

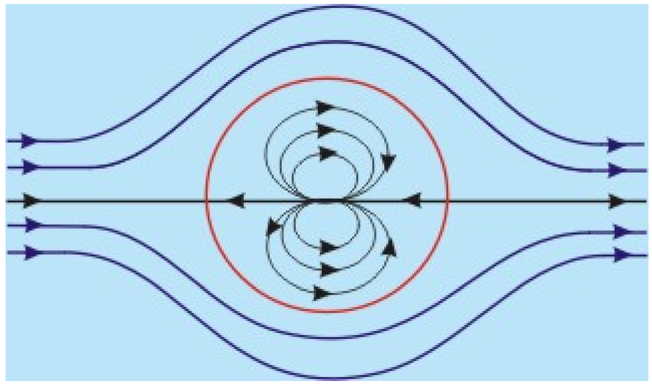


Fig. 22.3 Inviscid Flow past a Cylinder

Figure 22.3 shows the streamlines of the flow.

1. The **streamlines outside the circle** describe the **flow pattern of the inviscid irrotational flow across a cylinder.**
2. The **streamlines inside the circle may be disregarded** since this region is considered as a solid obstacle.

Flow Past a Source

When a uniform flow is added to that due to a source -

- fluid emitted from the source is swept away in the downstream direction
- stream function and velocity potential for this flow will be the sum of those for uniform flow and source

Stream function; $\psi = U_\infty r \sin\theta + \kappa\theta$

Velocity Potential; $\phi = U_\infty r \cos\theta + \kappa \ln r$

So $u_r = \dfrac{\partial \phi}{\partial r} = U_\infty \cos\theta + \dfrac{\kappa}{r}$

and $u_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} = \dfrac{1}{r}\left(-U_\infty r \sin\theta\right) = -U_\infty \sin\theta$

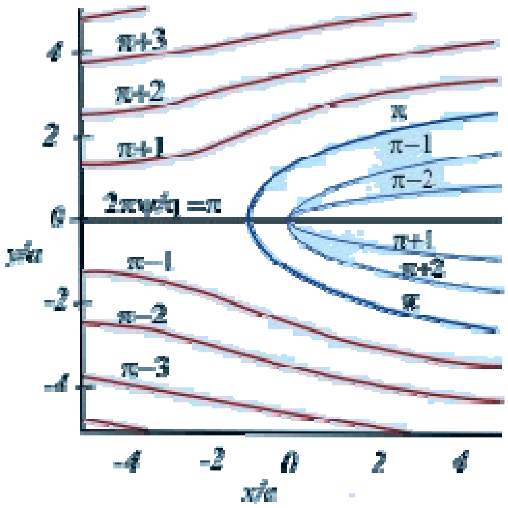


Fig 23.1 The streamlines of the flow past a line source for equal increments of 2πψ/q
The Plane coordinates are x/a, y/a where a=k/u

Explanation of Figure

- At the point x = -a, y = 0 fluid velocity is zero.
- This is called stagnation point of the flow
- Here the source flow is turned around by the oncoming uniform flow
- The parabolic streamline passing through stagnation point $\psi = \pm \pi k$ seperates uniform flow from the source flow.
- The streamline becomes parallel to x axis as $x \to \infty \quad \text{where } y = \pm \pi a$

Flow Past Vortex

when uniform flow is superimposed with a vortex flow -

- Flow will be asymmetrical about x - axis
- Again stream function and velocity potential will be the sum of those for uniform flow and vortex flow

Stream Function: $\psi = U_0 r \sin\theta - \frac{\Gamma}{2\pi} \ln r$

Velocity Potential: $\phi = U_0 r \cos\theta + \frac{\Gamma}{2\pi}\theta$

so that; $u_r = \frac{\partial \phi}{\partial r} = U_0 \cos\theta$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U_0 \sin\theta + \frac{\Gamma}{2\pi r}$$

Flow About a Rotating Cylinder

Magnus Effect

Flow about a rotating cylinder is equivalent to the combination of flow past a cylinder and a vortex.
As such in addition to superimposed uniform flow and a doublet, a vortex is thrown at the doublet centre which will simulate a rotating cylinder in uniform stream.

The pressure distribution will result in a force, a component of which will culminate in lift force

The phenomenon of generation of lift by a rotating object placed in a stream is known as Magnus effect.

Velocity Potential and Stream Function

The velocity potential and stream functions for the combination of doublet, vortex and uniform flow are

$$\phi = U_0 x + \frac{Z\cos\theta}{r} - \frac{\Gamma}{2\pi}\theta \quad \text{(clockwise rotation)} \qquad (23.1)$$

$$\psi = U_0 y - \frac{Z\sin\theta}{r} + \frac{\Gamma}{2\pi}\ln r \quad \text{(clockwise rotation)} \qquad (23.2)$$

By making use of either the stream function or velocity potential function, the velocity components are (putting x= rcosθ and y= rsinθ)

$$v_r = \frac{\partial\phi}{\partial r} = \left(U_0 - \frac{Z}{r^2}\right)\cos\theta \qquad (23.3)$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\left(U_0 + \frac{Z}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r} \qquad (23.4)$$

Stagnation Points

At the stagnation points the velocity components must vanish. From Eq. (23.3), we get

$$\cos\theta\left(U_0 - \frac{Z}{r^2}\right) = 0 \qquad (23.5)$$

Solution :

1. From Eq. (23.5) it is evident that a zero radial velocity component may occur at

- $\theta = \pm\frac{\pi}{2}$ and
- along the circle, $r = \left(\frac{\chi}{U_0}\right)^{1/2}$.

Eq. (23.4) depicts that a zero transverse velocity requires

$$\sin\theta = \frac{-\Gamma/2\pi r}{U_0 + \left(\frac{\chi}{r^2}\right)} \quad \text{or} \quad \theta = \sin^{-1}\left[\frac{-\Gamma/2\pi r}{U_0 + \frac{\chi}{r^2}}\right] \qquad (23.6)$$

At the stagnation point, both radial and transverse velocity components must be zero .

Thus the location of stagnation point occurs at

$$r = \left(\frac{\chi}{U_0}\right)^{1/2}$$

$$\theta = \sin^{-1}\left\{\frac{-\Gamma}{4\pi(\chi U_0)^{1/2}}\right\}$$

There will be two stagnation points since there are two angles for a given sine except for $\sin^{-1}(\pm 1)$

Determination of Stream Line

The streamline passing through these points may be determined by evaluating ψ at these points.

Substitution of the stagnation coordinate (r, θ) into the stream function (Eq. 23.2) yields

$$\psi = \left[U_0\left(\frac{\chi}{U_0}\right)^{1/2} - \frac{\chi}{\left(\frac{\chi}{U_0}\right)^{1/2}}\right]\sin\,\sin^{-1}\left[\frac{\Gamma}{4\pi(\chi U_0)^{1/2}}\right] + \frac{\Gamma}{2\pi}\ln\left(\frac{\chi}{U_0}\right)^{1/2}$$

$$\psi = \left[(U_0 \chi)^{1/2} - (U_0 \chi)^{1/2}\right]\left[\frac{-\Gamma}{4\pi(\chi U_0)^{1/2}}\right] + \frac{\Gamma}{2\pi}\ln\left(\frac{\chi}{U_0}\right)^{1/2}$$

or,

$$\psi_{stag} = \frac{\Gamma}{2\pi}\ln\left(\frac{\chi}{U_0}\right)^{1/2} \qquad (23.7)$$

Equating the general expression for stream function to the above constant, we get

$$U_0 r \sin\theta - \frac{\chi \sin\theta}{r} + \frac{\Gamma}{2\pi}\ln r = \frac{\Gamma}{2\pi}\ln\left(\frac{\chi}{U_0}\right)^{1/2}$$

By rearranging we can write

$$\sin\theta\left[U_0 r - \frac{\chi}{r}\right] + \frac{\Gamma}{2\pi}\left[\ln r - \ln\left(\frac{\chi}{U_0}\right)^{1/2}\right] = 0 \qquad (23.8)$$

All points along the circle $r = \left(\frac{\chi}{U_0}\right)^{1/2}$ **satisfy Eq. (23.8)** , since for this value of r, each quantity within parentheses in the equation is zero.

Considering the interior of the circle (on which $\psi = 0$) to be a solid cylinder, the outer streamline pattern is shown in Fig 23.2.

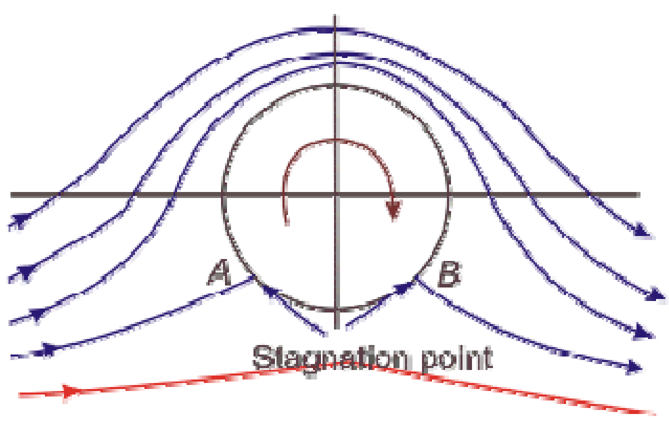


Fig 23.2 Flow Past a Cylinder with Circulation

At the stagnation point

$$\theta = \sin^{-1}\left[\frac{-(\Gamma/2\pi)}{2(\pi U_0)/r}\right]$$

$$\theta = \sin^{-1}\left[\frac{-\Gamma/2\pi}{2U_0 r}\right]$$

The limiting case arises for $\frac{(\Gamma/2\pi)}{U_0 r} = 2$, where $\theta = \sin^{-1}(-1) = -90^\circ$ and two stagnation points meet at the bottom as shown in Fig. 23.3.

In the case of a circulatory flow past the cylinder, the streamlines are symmetric with respect to the y-axis. The presures at the points on the cylinder surface are symmetrical with respect to the y-axis. There is no symmetry with respect to the x-axis. Therefore a resultant force acts on the cylinder in the direction of the y-axis, and the resultant force in the direction of the x-axis is equal to zero as in the flow without circulation; that is, the D'Alembert paradox takes place here as well. Thus, in the presence of circulation, different flow patterns can take place and, therefore, it is necessary for the uniqueness of the solution, to specify the magnitude of circulation.

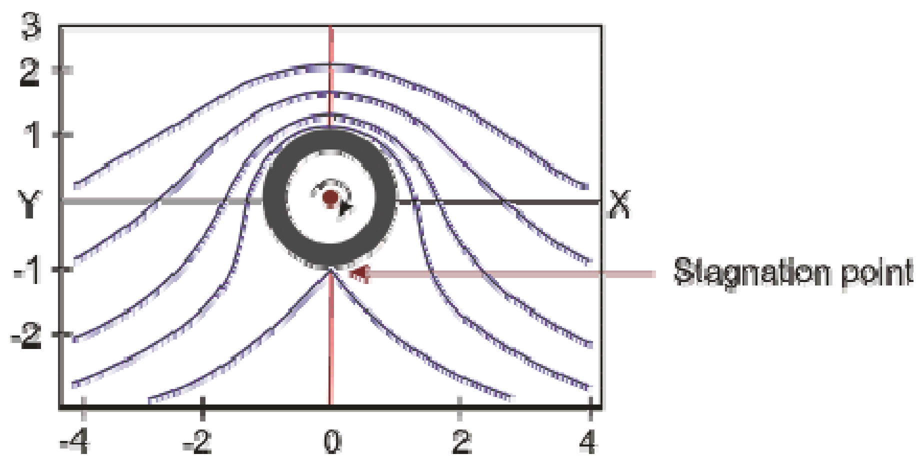


Fig 23.3 Flow Past a Circular Cylinder with Circulation Value $\frac{(\Gamma/2\pi)}{U_0 r} = 2$

However, in all these cases the **effects of the vortex and doublet become negligibly small as one moves a large distance from the cylinder.**

The flow is assumed to be uniform at infinity.

We have already seen that the change in strength G of the vortex changes the flow pattern, particularly the position of the stagnation points but the radius of the cylinder remains unchanged.

6.7 Non-Lifting and Lifting Flow over a Circular Cylinder:

Lift and Drag for Flow Past a Cylinder without Circulation

Pressure in the Cylinder Surface

Pressure becomes uniform at large distances from the cylinder (where the influence of doublet is small).

Let us imagine the pressure p_0 is known as well as uniform velocity U_0 . We can apply Bernoulli's equation between infinity and the points on the boundary of the cylinder.

Neglecting the variation of potential energy between the aforesaid point at infinity and any point on the surface of the cylinder, we can write

$$\frac{p_0}{\rho g}+\frac{U_0^2}{2g}=\frac{p_b}{\rho g}+\frac{U_b^2}{2g} \qquad (22.5)$$

where the subscript b represents the surface on the cylinder.

Since fluid cannot penetrate the solid boundary, the velocity **U_b should be only in the transverse direction** , or in other words, only v_θ component of velocity is present on the streamline $\psi = 0$.

Thus at $r=\left(\frac{\mathcal{I}}{U_0}\right)^{1/2}$

$$U_b=v_\theta\Big|_{at\ r=\left(\frac{\mathcal{I}}{U_0}\right)^{1/2}} = \frac{1}{2}\frac{\partial\phi}{\partial r}\Big|_{at\ r=\left(\frac{\mathcal{I}}{U_0}\right)^{1/2}} = -2U_0\sin\theta \qquad (22.6)$$

From eqs (22.5) and (22.6) we obtain

$$p_b = \rho g\left[\frac{U_0^2}{2g}+\frac{p_0}{\rho g}-\frac{(2U_0\sin\theta)^2}{2g}\right] \qquad (22.7)$$

Lift and Drag

Lift :force acting on the cylinder (per unit length) in the direction normal to uniform flow.

Drag: force acting on the cylinder (per unit length) in the direction parallel to uniform flow.

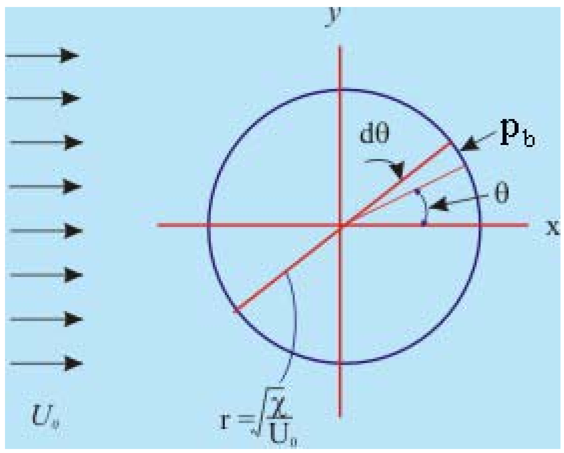


Fig 22.4 Calculation of Drag in a Cylinder

The drag is calculated by integrating the force components arising out of pressure, in the x direction on the boundary. Referring to Fig.22.4, the drag force can be written as

$$D=-\int_0^{2\pi} p_b\cos\theta\, r\,d\theta \qquad r\,d\theta = ds \rightarrow$$ infinitesimal length on the circumference

Since, $$r=\left(\frac{\chi}{U_0}\right)^{1/2}$$

$$D=-\int_0^{2\pi} p_b\cos\theta\left(\frac{\chi}{U_0}\right)^{1/2} d\theta$$

or, $$D=-\int_0^{2\pi}\rho g\left(\frac{\chi}{U_0}\right)^{1/2}\left[\frac{U_0^2}{2g}+\frac{p_0}{\rho g}-\frac{(2U_0\sin\theta)^2}{2g}\right]\cos\theta\, d\theta$$

$$D = -\int_0^{2\pi}\left[p_0 + \frac{\rho U_0^2}{2}\left(1 - 4U_0 \sin^2\theta\right)\right]\left(\frac{\chi}{U_0}\right)^{1/2}\cos\theta d\theta \qquad (22.8)$$

Similarly, the lift force may be calculated as

$$L = -\int_0^{2\pi} p_s \sin\theta \left(\frac{\chi}{U_0}\right)^{1/2} d\theta \qquad (22.9)$$

The Eqs (22.8) and (22.9) produce **D=0 and L=0 after** the integration is carried out.

However, **in reality, the cylinder will always experience some drag force. This contradiction between the inviscid flow result and the experiment is usually known as D 'Almbert paradox.**

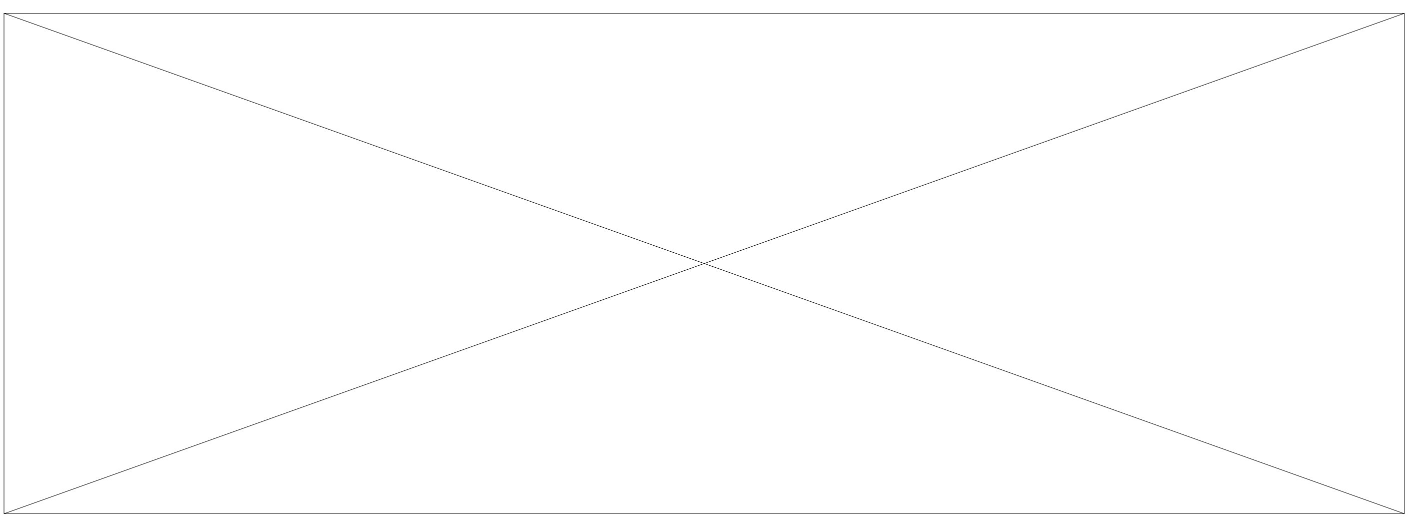
Bernoulli's equation can be used to calculate the pressure distribution on the cylinder surface

$$\frac{p_s(\theta)}{\rho g} - \frac{U_s^2(\theta)}{2g} = \frac{p_0}{\rho g} + \frac{U_0^2}{2g}$$

$$\frac{p_s(\theta) - p_0}{\rho} = \frac{U_0^2}{2}\left[1 - 4\sin^2\theta\right]$$

The pressure coefficient , c_p is therefore

$$C_p = \frac{p_s(\theta) - p_0}{\frac{1}{2}\rho U_0^2} = \left[1 - 4\sin^2\theta\right] \qquad (22.10)$$



Lift and Drag for Flow About a Rotating Cylinder

The pressure at large distances from the cylinder is uniform and given by p_0.

Deploying Bernoulli's equation between the points at infinity and on the boundary of the cylinder,

$$p_b = \rho g\left[\frac{U_0^2}{2g} + \frac{p_0}{\rho g} - \frac{U_b^2}{2g}\right] \qquad (23.9)$$

Hence,

$$U_b = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -2U_0 \sin\theta - \frac{r}{2\pi}\left[\frac{U_0}{x}\right]^{1/2} \qquad (23.10)$$

From Eqs (23.9) and (23.10) we can write

$$p_b = \rho g\left[\frac{U_0^2}{2g} + \frac{p_0}{\rho g}\right] - \left[\frac{-2U_0 \sin\theta - \frac{r}{2\pi}\left(\frac{U_0}{x}\right)^{1/2}}{2g}\right]^2 \qquad (23.11)$$

The lift may calculated as

$$L = -\int_0^{2\pi} p_b \sin\theta \left[\frac{x}{U_0}\right]^{1/2} d\theta$$

$$L = \int_0^{2\pi}\left\{\frac{\rho U_0^2}{2} + p_0 \quad \frac{\rho\left[-2U_0 \sin\theta - \frac{\Gamma}{2\pi}\left(\frac{U_0}{x}\right)^{1/2}\right]^2}{2}\right\}\left[\frac{x}{U_0}\right]^{1/2} (\sin\theta) d\theta$$

or,

$$L = -\int_0^{2\pi}\left[\frac{\rho U_0^2}{2}\left(\frac{x}{U_0}\right)^{1/2}\sin\theta + p_0\left(\frac{x}{U_0}\right)^{1/2}\sin\theta - \frac{\rho}{2}\left\{4U_0^2\sin^2\theta + \frac{4U_0\Gamma\sin\theta}{2\pi}\left(\frac{U_0}{x}\right)^{1/2} + \frac{\Gamma^2}{4\pi^2}\left[\frac{U_0}{x}\right.\right.\right.$$

$$L = -\int_0^{2\pi}\left[\frac{\rho U_0^2}{2}\left(\frac{x}{U_0}\right)^{1/2}\sin\theta + p_0\left(\frac{x}{U_0}\right)^{1/2}\sin\theta - 2\rho U_0^2\sin^3\theta\left(\frac{x}{U_0}\right)^{1/2} - \frac{\rho U_0\Gamma}{\pi}\sin^2\theta - \frac{\rho\Gamma^2}{8\pi^2}\left(\frac{x}{U_0}\right.\right.$$

$$L = \rho U_0 \Gamma \qquad (23.1$$

2)

The drag force , which includes the multiplication by cosθ (and integration over 2π) is zero.

- Thus the inviscid flow also demonstrates lift.
- lift becomes a simple formula involving only the density of the medium, free stream velocity and circulation.
- in two dimensional incompressible steady flow about a boundary of any shape, the lift is always a product of these three quantities.----- **Kutta- Joukowski theorem**

6.8 Pressure Distribution over Circular Cylinder in real flow:

Refer above Section

6.9 Kutta Joukowaski's Theorem:

Refer Above Section

6.10 Generation of Lift:

Refer Above Section

6.11 Lift on Airfoils:

Aerofoil Theory

Aerofoils are streamline shaped wings which are used in airplanes and turbo machinery. These shapes are such that the drag force is a very small fraction of the lift. The following nomenclatures are used for defining an aerofoil

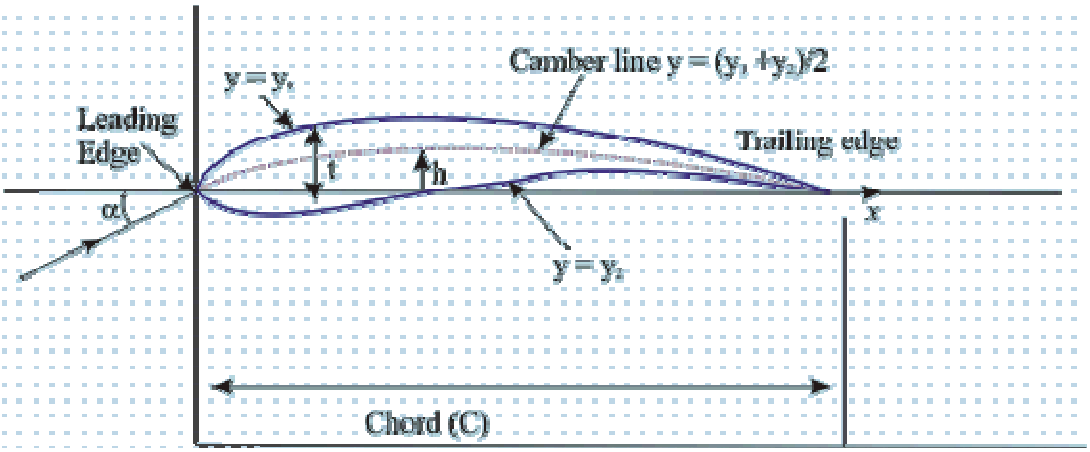


Fig 23.4 Aerofoil Section

- The **chord** (C) is the distance between the leading edge and trailing edge.
- The length of an aerofoil, normal to the cross-section (i.e., normal to the plane of a paper) is called the **span** of a aerofoil.
- The **camber line** represents the mean profile of the aerofoil. Some important geometrical parameters for an aerofoil are the ratio of maximum thickness to chord (t/C) and the ratio of maximum camber to chord (h/C). When these ratios are small, an aerofoil can be considered to be thin. For the analysis of flow, a thin aerofoil is represented by its camber.

The theory of thick cambered aerofoils uses a complex-variable mapping which transforms the inviscid flow across a rotating cylinder into the flow about an aerofoil shape with circulation.

Flow Around a Thin Aerofoil

- Thin aerofoil theory is based upon the superposition of uniform flow at infinity and a continuous distribution of clockwise free vortex on the camber line having circulation density $\gamma(s)$ per unit length .
- The circulation density $\gamma(s)$ should be such that the resultant flow is tangent to the camber line at every point.
- Since the slope of the camber line is assumed to be small, $\gamma(s)ds \approx \gamma(\xi)d\xi$. The total circulation around the profile is given by

$$\Gamma = \int_0^C \gamma(\eta)\, d\eta \qquad (23.13)$$

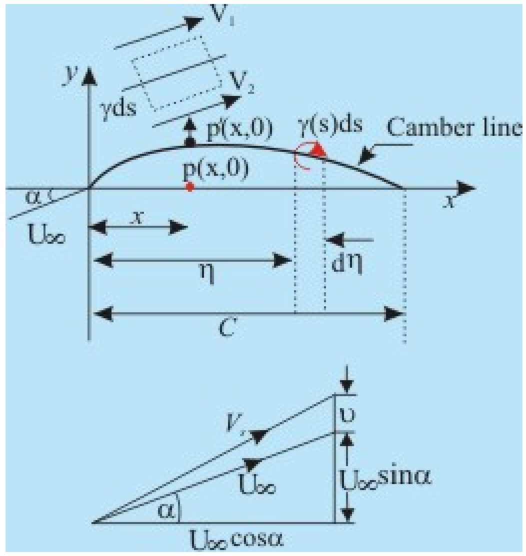


Fig 23.5 Flow Around Thin Aerofoil

A vortical motion of strength $\gamma d\eta$ at x= η develops a velocity at the **point p** which may be expressed as

$$du = \frac{\gamma(\eta)d\eta}{2\pi(\eta - x)} \text{ acting upwards}$$

The total induced velocity in the upward direction at **point p** due to the entire vortex distribution along the camber line is

$$u(x) = \frac{1}{2\pi}\int_0^C \frac{\gamma(\eta)d\eta}{(\eta - x)} \qquad (23.14)$$

For a small camber (having small α), this expression is identically valid for the induced velocity at **point p'** due to the vortex sheet of variable strength $\gamma(s)$ on the camber line. The resultant velocity due to U_∞ and v(x) must be tangential to the camber line so that the slope of a camber line may be expressed as

$$\frac{dy}{dx} = \frac{U_\infty \sin\alpha + u}{U_\infty \cos\alpha} = \tan\alpha + \frac{u}{U_\infty \cos\alpha}$$

$$\frac{dy}{dx} = \alpha + \frac{v}{U_\infty}\left[\text{since } \alpha \text{ is very small}\right] \qquad (23.15)$$

From Eqs (23.14) and (23.15) we can write

$$\frac{dy}{dx} = \alpha - \frac{1}{2\pi U_\infty}\int_0^c \frac{\gamma(\eta)d\eta}{\eta - x}$$

Consider an element ds on the camber line. Consider a small rectangle (drawn with dotted line) around ds. The upper and lower sides of the rectangle are very close to each other and these are parallel to the camber line. The other two sides are normal to the camber line. The circulation along the rectangle is measured in clockwise direction as

$V_1 ds - V_2 ds = \gamma ds$ [normal component of velocity at the camber line should be zero]

or $V_1 - V_2 = \gamma$

If the mean velocity in the tangential direction at the camber line is given by $V_s = (V_1 + V_2)/2$, it can be rewritten as

$V_1 = V_s + \frac{\gamma}{2}$ and $V_2 = V_s - \frac{\gamma}{2}$

if v is very small $[v << U_\infty]$, V_s becomes equal to U_∞. The difference in velocity across the camber line brought about by the vortex sheet of variable strength $\gamma(s)$ causes pressure difference and generates lift force.

Generation of Vortices Around a Wing

- **The lift around an aerofoil is generated following Kutta-Joukowski theorem** . Lift is a product of ρ , U_∞ and the circulation Γ.

  $Lift = \rho U_\infty \Gamma$

- When the motion of a wing starts from rest, vortices are formed at the trailing edge.
- At the start, there is a velocity discontinuity at the trailing edge. This is eventual because near the trailing edge, the velocity at the bottom surface is higher than that at the top surface. This discrepancy in velocity culminates in the formation of vortices at the trailing edge.
- Figure 23.6(a) depicts the formation of starting vortex by impulsively moving aerofoil. However, the starting vortices induce a counter circulation as shown in Figure 23.6(b). The circulation around a path (ABCD) enclosing the wing and just shed (starting) vortex must be zero. Here we refer to **Kelvin's theorem** once again.

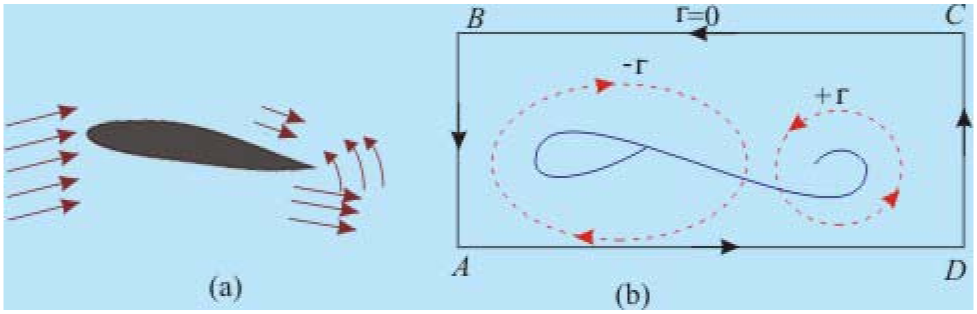


Fig 23.6 Vortices Generated when an Aerofoil Just Begins to Move

- Initially, the flow starts with the zero circulation around the closed path. Thereafter, due to the change in angle of attack or flow velocity, if a fresh starting vortex is shed, the circulation around the wing will adjust itself so that a net zero vorticity is set around the closed path.
- Real wings have finite span or finite aspect ratio (AR) λ , defined as

$$\lambda = \frac{b^2}{A_s} \qquad (23.16)$$

where b is the span length, A_s is the plan form area as seen from the top..

- For a wing of finite span, the end conditions affect both the lift and the drag. In the leading edge region, pressure at the bottom surface of a wing is higher than that at the top surface. The longitudinal vortices are generated at the edges of finite wing owing to pressure differences between the bottom surface directly facing the flow and the top surface.

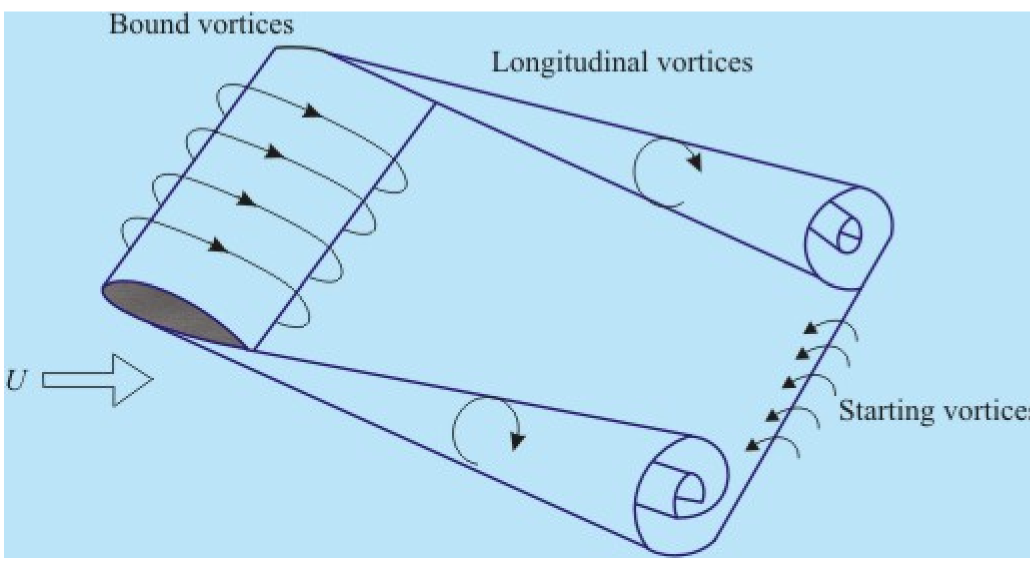


Fig 23.7 Vortices Around a Finite Wing

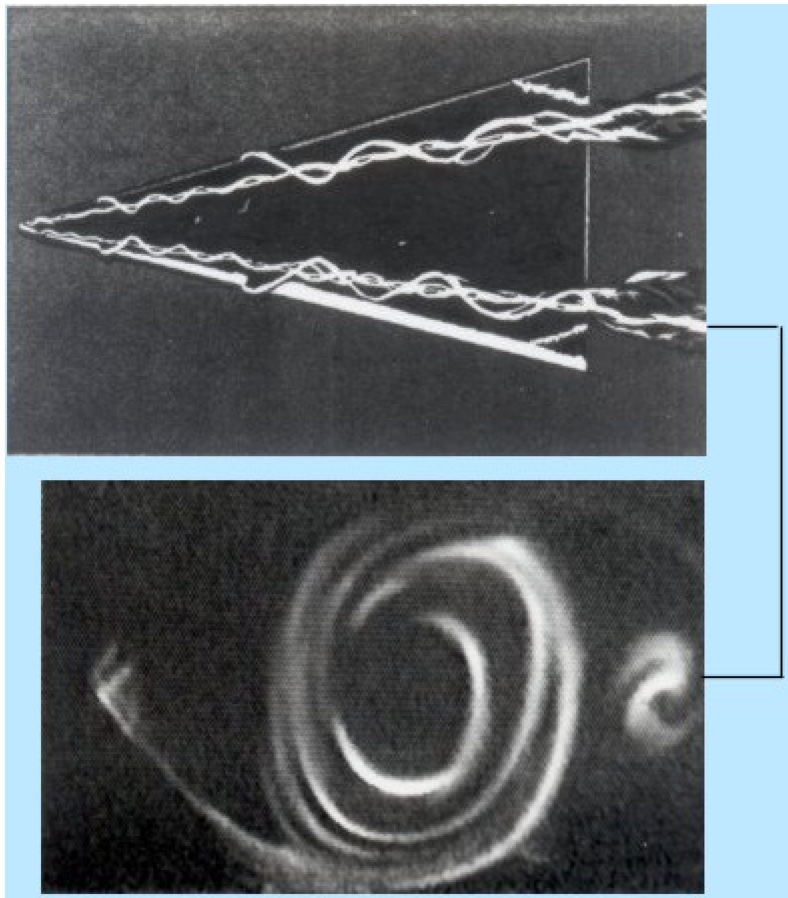


Fig 23.8 Generation of Longitudinal Vortices

Chapter 7
Introduction to Viscous Flows

7.1 Qualitative Aspects of Viscous Flows:

General Viscosity Law

Newton's viscosity law is

$$\tau = \mu \frac{\partial V}{\partial n} \qquad (24.1)$$

where,

τ = Shear Stress,
n is the coordinate direction normal to the solid-fluid interface,
μ is the coefficient of viscosity, and
V is velocity.

The above law is valid for parallel flows.

Considering Stokes' viscosity law: **shear stress is proportional to rate of shear strain** so that

$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \qquad (24.2a)$$

$$\tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] \qquad (24.2b)$$

$$\tau_{zx} = \tau_{xz} = \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \qquad (24.2c)$$

τ **has two subscripts---**

first subscript : denotes the direction of the normal to the plane on which the stress acts, while the
second subscript : denotes direction of the force which causes the stress.

The expressions of Stokes' law of viscosity for normal stresses are

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} + \mu'\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] \quad (24.3a)$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial \upsilon}{\partial y} + \mu'\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] \quad (24.3b)$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z} + \mu'\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] \quad (24.3c)$$

where μ' is a proportionality factor and it is related to the second coefficient of viscosity μ_1 by the relationship $\mu_1 = \mu' + \frac{2}{3}\mu$.

We have already seen that the thermodyamic pressure is $p = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$
Now if we add the three equations 24.3(a),(b) and (c) , we obtain,

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p + 2\mu\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] + 3\mu'\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right]$$

or

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p + (2\mu + 3\mu')\left[\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right] \quad (24.4)$$

- For incompressible fluids, $\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{V} = 0$
  So, $p = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$ is satisfied eventually. This is known as **Thermodynamic pressure**.
- For compressible fluids, **Stokes' hypothesis** is $\mu' + \frac{2}{3}\mu = 0$.
- Invoking this to Eq. (24.4), will finally result in $p = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$ (same as for incompressible fluid).
- Interesting historical aspects of the Stoke's assumption $3\mu' + 2\mu = 0$ can be found in Truesdell (1952)[1] .
- --

† Truesdell , C.A. "Stoke's Principle of Viscosity", Journal of Rational Mechanics and Analysis, Vol.1, pp.228-231,1952.

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- Generally, **fluids obeying the ideal gas equation follow this hypothesis and they are called Stokesian fluids** .
- The second coefficient of viscosity, μ_I has been verified to be negligibly small. Substituting μ for μ' in 24.3a, 24.3b, 24.3c we obtain

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] \quad (24.5a)$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] \quad (24.5b)$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] \quad (24.5c)$$

In deriving the above stress-strain rate relationship, it was assumed that a fluid has the following properties

- Fluid is homogeneous and isotropic, i.e. the relation between components of stress and those of rate of strain is the same in all directions.
- Stress is a linear function of strain rate.
- The stress-strain relationship will hold good irrespective of the orientation of the reference coordinate system.

The stress components must reduce to the hydrostatic pressure "**p**" (**typically** thermodynamic pressure = hydrostatic pressure) when all the gradients of velocities are zero.

7.2 Viscosity and Thermal Conductivity:

Refer Section 1.3

7.3 Phenomenon of Separation:

Seperation of Boundary Layer

- It has been observed that the **flow is reversed at the vicinity of the wall** under certain conditions.
- The phenomenon is termed as **separation of boundary layer**.
- Separation takes place **due to excessive momentum loss near the wall in a boundary layer trying to move downstream against increasing pressure, i.e.,** $\frac{dp}{dx} > 0$**, which is called** ***adverse pressure gradient*****.**
- Figure 29.2 shows the flow past a circular cylinder, in an infinite medium.
    1. Up to $\theta = 90^\circ$, the flow area is like a constricted passage and the flow behaviour is like that of a nozzle.
    2. Beyond $\theta = 90^\circ$ the flow area is diverged, therefore, the flow behaviour is much similar to a diffuser

This dictates the inviscid pressure distribution on the cylinder which is shown by a firm line in Fig. 29.2.

Here

p_∞ **: pressure in the free stream**

U_∞ **: velocity in the free stream and**

p***:*** **is the local pressure on the cylinder.**

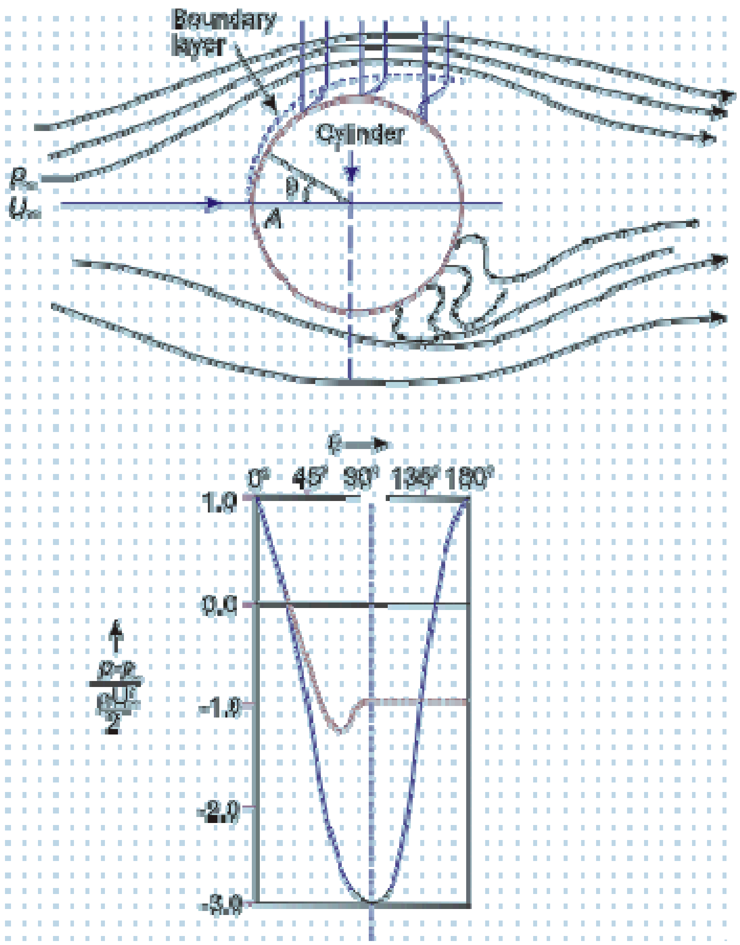


Fig. 29.2 Flow separation and formation of wake behind a circular cylinder

- Consider the forces in the flow field.
  In the **inviscid region,**
  1. **Until** $\theta = 90^\circ$ the pressure force and the force due to streamwise acceleration i.e. inertia forces are acting in the same direction (**pressure gradient** being **negative/favourable**)
  2. **Beyond** $\theta = 90^\circ$, the **pressure gradient is positive or adverse**. Due to the adverse pressure gradient the pressure force and the force due to acceleration will be opposing each other in the in viscid zone of this part.

- So long as no viscous effect is considered, the situation does not cause any sensation.
  In the **viscid region** (near the solid boundary),

  3. **Up to** $\theta = 90^\circ$**, th**e viscous force opposes the combined pressure force and the force due to acceleration. Fluid particles overcome this viscous resistance **due to continuous conversion of pressure force into kinetic energy.**

4. Beyond $\theta = 90^\circ$, within the viscous zone, the flow structure becomes different. It is seen that the force due to acceleration is opposed by both the viscous force and pressure force.

- Depending upon the magnitude of adverse pressure gradient, **somewhere around $\theta = 90^\circ$, the fluid particles, in the boundary layer are separated from the wall** and driven in the upstream direction. However, the far field external stream pushes back these separated layers together with it and develops **a broad pulsating wake behind the cylinder**.
- **The mathematical explanation of flow-separation :** The point of separation may be defined as the limit between forward and reverse flow in the layer very close to the wall, i.e., at the point of separation

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \qquad (29.16)$$

This means that the shear stress at the wall, $\tau_w = 0$. But at this point, the adverse pressure continues to exist and at the downstream of this point the flow acts in a reverse direction resulting in a back flow.

- We can also explain flow separation using the argument about the second derivative of velocity u at the wall. From the dimensional form of the momentum at the wall, where $u = v = 0$, we can write

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \frac{1}{\mu}\frac{dp}{dx} \qquad (29.17)$$

- Consider the situation due to a **favourable pressure gradient** where $\frac{dp}{dx} < 0$ we have,

1. $(\partial^2 u/\partial y^2)_{wall} < 0$. (From Eq. (29.17))
2. **As we proceed towards the free stream, the velocity u approaches U_∞ asymptotically,** so $\partial u/\partial y$ decreases at a continuously lesser rate in y direction.
3. This means that $\partial^2 u/\partial y^2$ remains less than zero near the edge of the boundary layer.
4. The curvature of a velocity profile $\partial^2 u/\partial y^2$ is always negative as shown in (Fig. 29.3a)

- Consider the case of **adverse pressure gradient**, $\partial p/\partial x > 0$
  1. At the boundary, the curvature of the profile must be positive (since $\partial p/\partial x > 0$).

2. Near the interface of boundary layer and free stream the previous argument regarding $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ still holds good and the curvature is negative.
3. Thus we observe that for an adverse pressure gradient, there must exist a point for which $\partial^2 u/\partial y^2 = 0$. This point is known as *point of inflection* of the velocity profile in the boundary layer as shown in Fig. 29.3b
4. However, point of separation means $\partial u/\partial y = 0$ at the wall.
5. $\partial^2 u/\partial y^2 > 0$ at the wall since separation can only occur due to adverse pressure gradient. But we have already seen that at the edge of the boundary layer, $\partial^2 u/\partial y^2 < 0$. It is therefore, clear that **if there is a point of separation, there must exist a point of inflection in the velocity profile.**

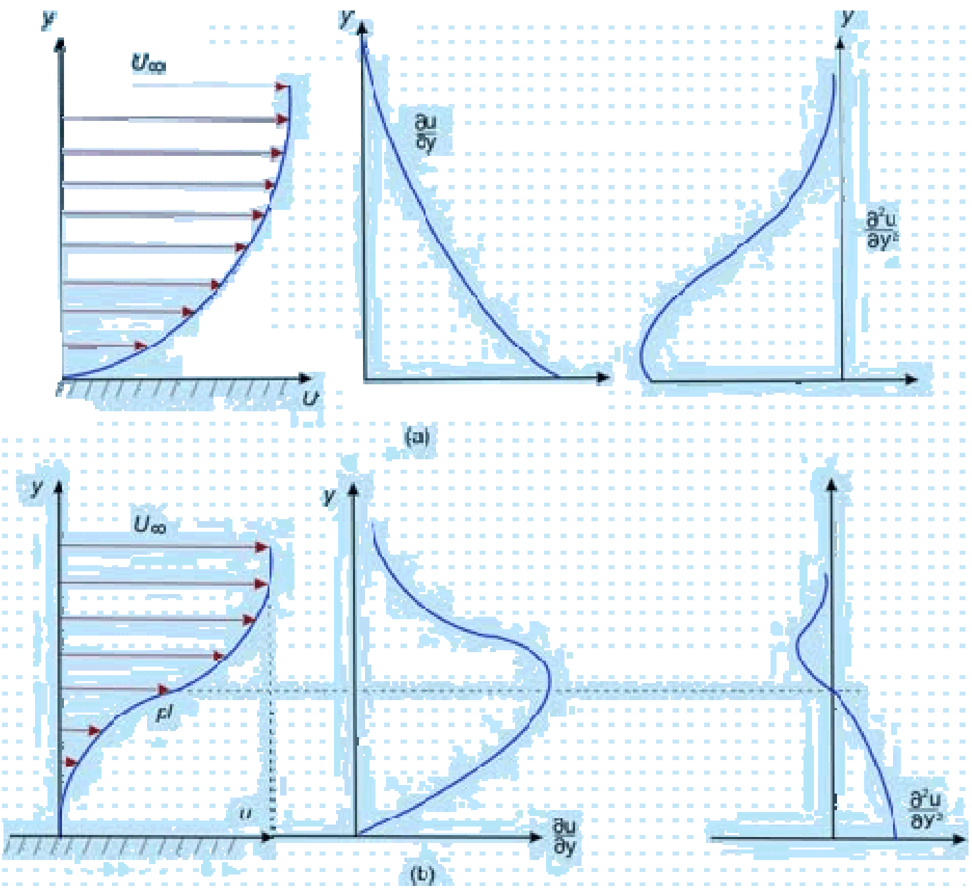


Fig. 29.3 Velocity distribution within a boundary layer

(a) Favourable pressure gradient, $\frac{dp}{dx} < 0$

(b) adverse pressure gradient, $\frac{dp}{dx} > 0$

1. Let us reconsider the flow past a circular cylinder and continue our **discussion on the wake behind a cylinder.** The pressure distribution which was shown by the firm line in

Fig. 21.5 is obtained from the potential flow theory. However. somewhere near $\theta = 90^\circ$ **(in experiments it has been observed to be at $\theta = 81^\circ$) . the boundary layer detaches itself from the wall.**

2. Meanwhile, **pressure in the wake remains close to separation-point-pressure** since the eddies (formed as a consequence of the retarded layers being carried together with the upper layer through the action of shear) cannot convert rotational kinetic energy into pressure head. The actual pressure distribution is shown by the dotted line in Fig. 29.3.
3. Since the **wake zone pressure is less than that of the forward stagnation point** (pressure at point A in Fig. 29.3), the cylinder experiences a drag force which is basically attributed to the pressure difference.

   The drag force, brought about by the pressure difference is known as *form drag* whereas the shear stress at the wall gives rise to *skin friction drag*. Generally, these two drag forces together are responsible for resultant drag on a body

7.4 Navier Stokes Equation in Vector Form:

A general way of deriving the Navier-Stokes equations from the basic laws of physics.

- Consider a general flow field as represented in Fig. 25.1.
- Imagine a closed **control volume,** V_0 within the flow field. The control volume is **fixed in space** and the fluid is moving through it. The control volume occupies reasonably large finite region of the flow field.
- A **control surface , A_0** is defined as the surface which **bounds** the volume V_0 .
- According to **Reynolds transport theorem, "**The rate of change of momentum for a **system** equals the sum of the rate of change of momentum inside the **control volume** and the rate of **efflux** of momentum across the **control surface"**.
- **The rate of change of momentum for a system** (in our case, the control volume boundary and the system boundary are same) **is equal to the net external force acting on it.**

Now, we shall transform these statements into equation by accounting for each term,

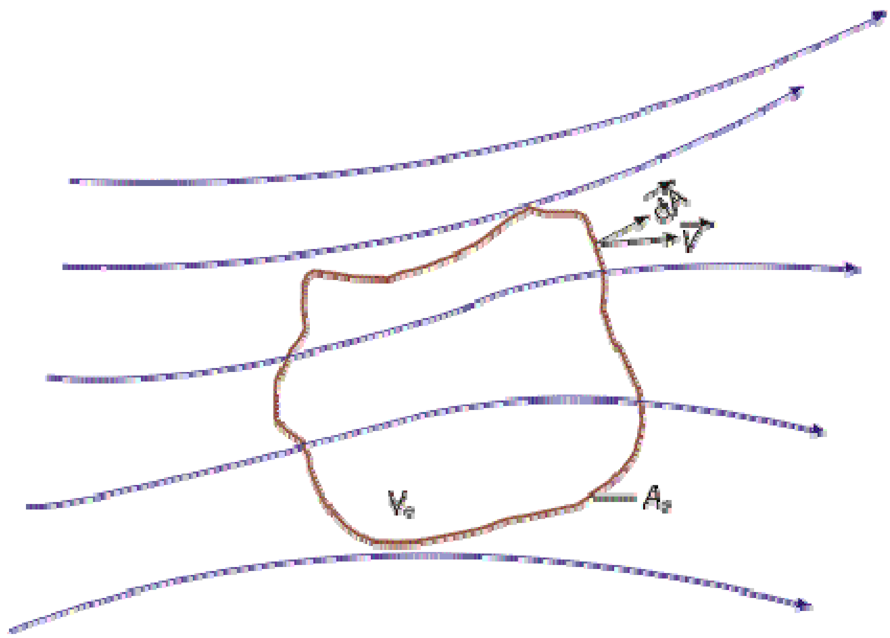


FIG 25.1 Finite control volume fixed in space with the fluid moving through it

- **Rate of change of momentum inside the control volume**

$$= \frac{\partial}{\partial t}\iiint_{V_0} \rho \vec{V}\, dV$$

$$= \iiint_{V_0} \frac{\partial}{\partial t}(\rho \vec{V})\, dV \qquad (25.1)$$

(since t is independent of space variable)

- **Rate of efflux of momentum through control surface**

$$\iint_{A_0} \rho \vec{V}(\vec{V}\cdot d\vec{A}) = \iint_{A_0} \rho \vec{V}\vec{V}\cdot\hat{n}\, dA$$

$$= \iiint_{V_0} \left(\vec{V}(\nabla\cdot\rho\vec{V}) + \rho\vec{V}\cdot\nabla\vec{V}\right) dV \qquad (25.2)$$

- **Surface force acting on the control volume**

$$= \iint_{A_0} d\vec{A}\cdot\sigma$$

(σ is symmetric stress tensor)

$$= \iiint_{V_0} (\nabla\cdot\sigma)\, dV \qquad (25.3)$$

- **Body force acting on the control volume**

$$\iiint_{V_0} \rho \vec{f}_b \, dV \qquad (25.4)$$

$\vec{f}_b$ in Eq. (25.4) is the body force per unit mass.

- **Finally, we get,**

$$\iiint_{V_0} \left(\frac{\partial}{\partial t}(\rho \vec{V}) + \vec{V}(\nabla.\rho\vec{V}) + \rho\vec{V}.\nabla\vec{V} \right) dV$$
$$= \iiint_{V_0} \left(\nabla.\sigma + \rho\vec{f}_b \right) dV$$

or

or, $\rho \dfrac{\partial \vec{V}}{\partial t} + \vec{V}\dfrac{\partial \rho}{\partial t} + \rho\vec{V}.\nabla\vec{V} + \vec{V}(\nabla.\rho\vec{V}) = \nabla.\sigma + \rho\vec{f}_b$

or $\rho\left(\dfrac{\partial \vec{V}}{\partial t} + \vec{V}.\nabla\vec{V} \right) + \vec{V}\left(\dfrac{\partial \rho}{\partial t} + \nabla.\rho\vec{V} \right) = \nabla.\sigma + \rho\vec{f}_b \qquad (25.5)$

We know that $\dfrac{\partial \rho}{\partial t} + \nabla.\rho\vec{V} = 0$ is the general form of **mass conservation equation** (popularly known as the **continuity equation**), valid for both **compressible** and **incompressible** flows.

- Invoking this relationship in Eq. (25.5), we obtain

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}.\nabla\vec{V} \right) = \nabla.\sigma + \rho\vec{f}_b$$

or $\rho\dfrac{D\vec{V}}{Dt} = \nabla.\sigma + \rho\vec{f}_b \qquad (25.6)$

- Equation (25.6) is referred to as **Cauchy's equation of motion** . In this equation, σ is the stress tensor,
- After having substituted σ we get

$$\nabla.\sigma = -\nabla p + (\mu' + \mu)\nabla(\nabla.\vec{V}) + \mu\nabla^2\vec{V} \qquad (25.8)$$

From **Stokes's hypothesis** we get, $\mu' + \dfrac{2}{3}\mu = 0 \qquad (25.9)$

Invoking above two relationships into Eq.(25.6) we get

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \mu \nabla^2 \vec{V} + \frac{1}{3}\mu \nabla\left(\nabla \cdot \vec{V}\right) + \rho \vec{f}_b \qquad (25.10)$$

This is the most general form of Navier-Stokes equation.

7.5 Viscous Flow Energy Equation: Under Development

7.6 Exact Solutions to Navier Stokes Equations:

Exact Solutions Of Navier-Stokes Equations

Consider a class of flow termed as parallel flow in which **only one velocity term is nontrivial** and all the fluid particles move in one direction only.

- We **choose** x **to be the direction along which all fluid particles travel** , i.e. $u \neq 0, v = w = 0$. Invoking this in continuity equation, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}^{0} + \frac{\partial w}{\partial z}^{0} = 0$$

$$\frac{\partial u}{\partial x} = 0 \text{ which means } u = u(y, z, t)$$

- Now. Navier-Stokes equations for incompressible flow become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}^{0} + v\frac{\partial u}{\partial y}^{0} + w\frac{\partial u}{\partial z}^{0} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2}^{0} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\frac{\partial v}{\partial t}^{0} + u\frac{\partial v}{\partial x}^{0} + v\frac{\partial v}{\partial y}^{0} + w\frac{\partial v}{\partial z}^{0} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2}^{0} + \frac{\partial^2 v}{\partial y^2}^{0} + \frac{\partial^2 v}{\partial z^2}^{0}\right]$$

$$\cancelto{0}{\frac{\partial v}{\partial t}} + u\cancelto{0}{\frac{\partial v}{\partial x}} + v\cancelto{0}{\frac{\partial v}{\partial y}} + w\cancelto{0}{\frac{\partial v}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\cancelto{0}{\frac{\partial^2 v}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial z^2}}\right]$$

So, we obtain

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \text{ which means } p = p(x) \text{ alone}$$

and $$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \qquad (25.11)$$

7.7 Plane Poiseuille Flow:

Parallel Flow in a Straight Channel

Consider **steady flow between two infinitely broad parallel plates** as shown in Fig. 25.2.

Flow is independent of any variation in z direction, hence, z dependence is gotten rid of and Eq. (25.11) becomes

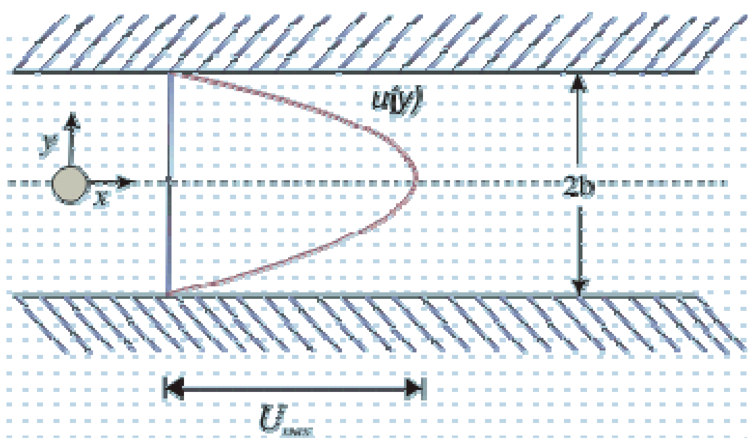


FIG 25.2 Parallel flow in a straight channel

$$\frac{dp}{dx} = \mu\frac{\partial^2 u}{\partial y^2} \qquad (25.12)$$

The boundary conditions are at y = b, u = 0; and y = -b, u = O.

- From Eq. (25.12), we can write

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$$

or $$u = \frac{1}{2\mu}.\frac{dp}{dx}y^2 + C_1 y + C_2$$

- Applying the boundary conditions, the constants are evaluated as

$C_1 = 0$ and $C_2 = -\frac{1}{\mu}.\frac{dp}{dx}.\frac{b^2}{2}$

So, the solution is

$$u = -\frac{1}{2\mu}.\frac{dp}{dx}\left(b^2 - y^2\right) \qquad (25.13)$$

which implies that the velocity profile is parabolic.

Average Velocity and Maximum Velocity

- To establish the relationship between the maximum velocity and average velocity in the channel, we analyze as follows

  At y = 0, $u = U_{max}$; this yields

$$U_{max} = -\frac{b^2}{2\mu}\frac{dp}{dx} \qquad (25.14a)$$

On the other hand, the average velocity,

$$U_{av} = \frac{Q}{2b} = \frac{flow\ rate}{flow\ area} = \frac{1}{2b}\int_{-b}^{b} u\,dy$$

or $$U_{av} = \frac{2}{2b}\int_0^b -\frac{1}{2\mu}.\frac{dp}{dx}\left(b^2 - y^2\right)dy$$

$$= \frac{1}{2\mu}\frac{dp}{dx}.\frac{1}{b}\left\{\left[b^2 y\right]_0^b - \left(\frac{y^3}{3}\right)_0^b\right\}$$

Finally, $$U_{av} = -\frac{1}{2\mu}\frac{dp}{dx}.\frac{2}{3}b^2 \qquad (25.14b)$$

So, $\frac{U_{av}}{U_{max}} = \frac{2}{3}$ or $U_{max} = \frac{3}{2}U_{av}$ (25.14c)

- The shearing stress at the wall for the parallel flow in a channel can be determined from the velocity gradient as

$$\tau_{yx}|_b = \mu\left(\frac{\partial u}{\partial y}\right)_b = b\frac{dp}{dx} = -2\mu\frac{U_{max}}{b}$$

Since the upper plate is a "minus y surface", a negative stress acts in the positive x direction, i.e. to the right.

- The local friction coefficient, C_f is defined by

$$C_f = \frac{\left|\left(\tau_{yx}\right)_b\right|}{\frac{1}{2}\rho U_{av}^2} = \frac{3\mu U_{av}/b}{\frac{1}{2}\rho U_{av}^2}$$

$$C_f = \frac{12}{\frac{\rho U_{av}(2b)}{\mu}} = \frac{12}{Re} \qquad (25.14d)$$

where $Re = U_{av}(2b)/\nu$ is the Reynolds number of flow based on average velocity and the channel height (2b).

- Experiments show that Eq. (25.14d) is valid in the laminar regime of the channel flow.
- The maximum Reynolds number value corresponding to fully developed laminar flow, for which a stable motion will persist, is 2300.
- In a reasonably careful experiment, laminar flow can be observed up to even Re = 10,000.
- But the value below which the flow will always remain laminar, i.e. **the critical value of Re is 2300.**

7.8 Couette Flow:

Couette Flow

Couette flow is the flow between **two parallel plates** (Fig. 26.1). Here, one plate is at **rest** and the other is **moving** with a velocity *U* . Let us assume the plates are infinitely large in *z* direction, so the ***z* dependence is not there**.

The **governing equation** is

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2}$$

flow is independent of any variation in z-direction.
The **boundary conditions** are ---(i)**At y = 0, u = 0** (ii)**At y = h, u = U.**

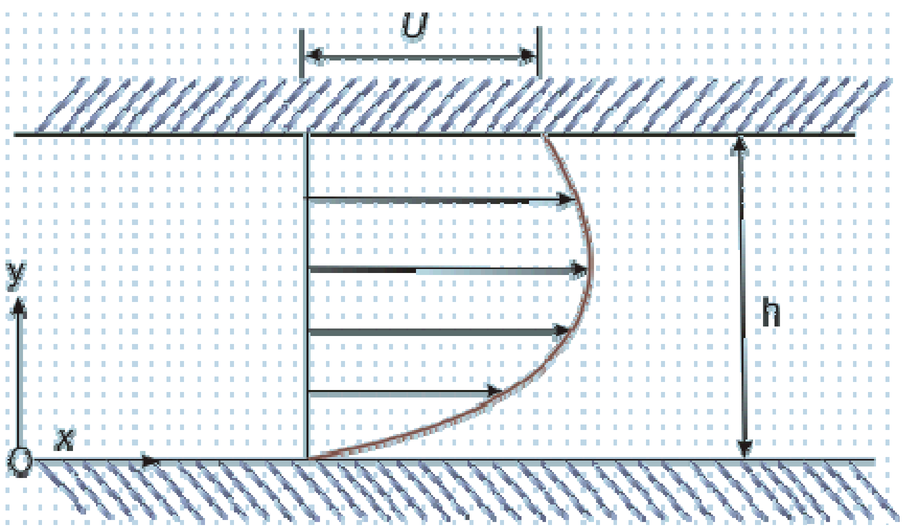


FIG 26.1 Couette flow between two parallel flat plates

- We get,

$$u = \frac{1}{2\mu} \cdot \frac{dp}{dx} y^2 + C_1 y + C_2$$

Invoking the condition (**at y = 0, u = 0**), C_2 becomes equal to zero.

$$u = \frac{1}{2\mu} \cdot \frac{dp}{dx} y^2 + C_1 y$$

Invoking the other condition (**at y = h, u = U**),

$$C_1 = \frac{U}{h} - \frac{1}{2\mu}\cdot\frac{dp}{dx}h$$

So, $$u = \frac{y}{h}U - \frac{h^2}{2\mu}\cdot\frac{dp}{dx}\cdot\frac{y}{h}\left(1-\frac{y}{h}\right) \qquad (26.1)$$

Equation (26.1) can also be expressed in the form

$$\frac{u}{U} = \frac{y}{h} - \frac{h^2}{2\mu U}\cdot\frac{dp}{dx}\cdot\frac{y}{h}\left(1-\frac{y}{h}\right)$$

or, $$\frac{u}{U} = \frac{y}{h} + P\frac{y}{h}\left(1-\frac{y}{h}\right) \qquad (26.2a)$$

Where

$$P = -\frac{h^2}{2\mu U}\left(\frac{dp}{dx}\right),$$

Equation (26.2a) describes the velocity distribution in non-dimensional form across the channel with **P** as a parameter known as the **non-dimensional pressure gradient .**

- When **P = 0**, the velocity distribution across the channel is reduced to

$$\frac{u}{U} = \frac{y}{h}$$

This particular case is known as **simple Couette flow.**

- When **P > 0** , i.e. for a ***negative*** or ***favourable*** **pressure gradient** $(-dp/dx)$ in the direction of motion, the velocity is positive over the whole gap between the channel walls. For negative value of P (**P < 0**), there is a ***positive*** or ***adverse*** **pressure gradient** in the direction of motion and the velocity over a portion of channel width can become negative and **back flow may occur** near the wall which is at rest. **Figure 26.2a** shows the effect of dragging action of the upper plate exerted on the fluid particles in the channel for different values of pressure gradient.

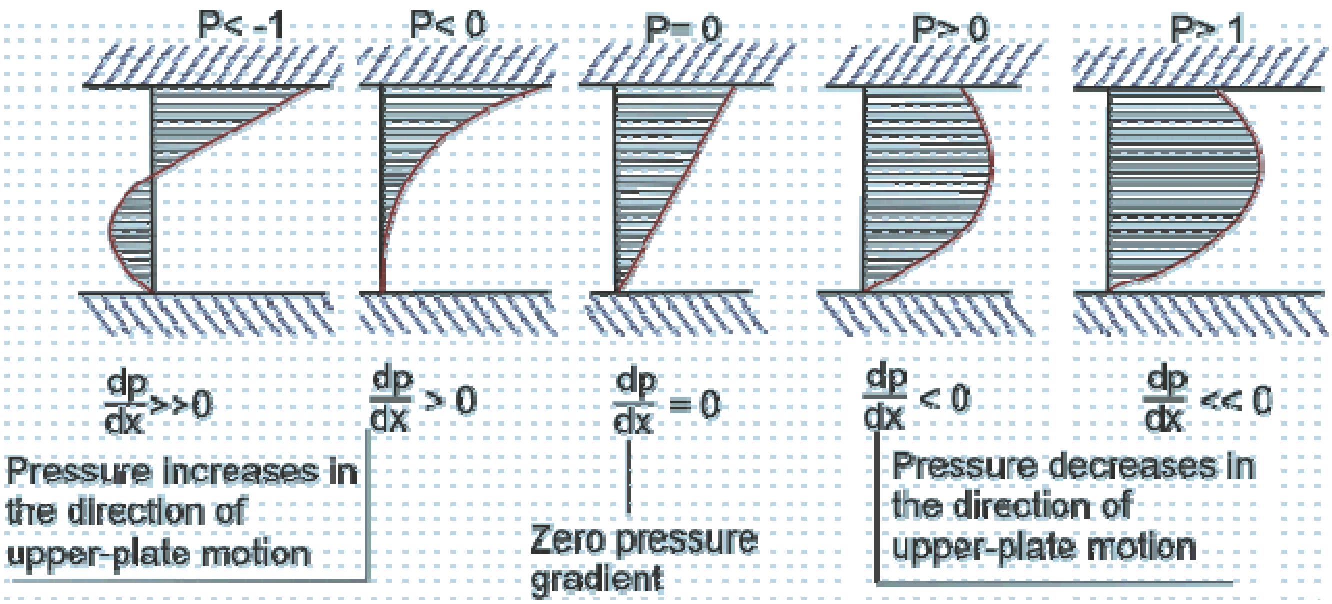


FIG 26.2a - Velocity profile for the Couette flow for various values of pressure gradient

Maximum and minimum velocities

The quantitative description of non-dimensional velocity distribution across the channel, depicted by Eq. (26.2a), is shown

in **Fig. 26.2b**.

- The location of maximum or minimum velocity in the channel is found out by setting du/dy **=0**. From Eq. (26.2a), we can write

$$\frac{du}{dy} = \frac{U}{h} + \frac{PU}{h}\left(1 - 2\frac{y}{h}\right)$$

Setting $\frac{du}{dy} = 0$ gives

$$\frac{y}{h} = \frac{1}{2} + \frac{1}{2P} \qquad 26.2b$$

- It is interesting to note that **maximum velocity for P = 1 occurs at y/h = 1 and equals to** ***U*** **.** For **P > 1**, the maximum velocity occurs at a location $\frac{y}{h} < 1$ **.**
- This means that with P > 1, the fluid particles attain a velocity higher than that of the moving plate at a location somewhere below the moving plate.

- For **P = -1**, the minimum velocity occurs, at $\frac{y}{h} = 0$. For **P < -1**, the minimum velocity occurs at a location $\frac{y}{h} > 1$.
- This means that there occurs a back flow near the fixed plate. The values of maximum and minimum velocities can be determined by substituting the value of *y* from Eq. (26.2b) into Eq. (26.2a) as

$$u_{max} = \frac{U(1+P)^2}{4P} \qquad for\ P \geq 1$$

$$u_{min} = \frac{U(1+P)^2}{4P} \qquad for\ P \leq 1 \qquad (26.2c)$$

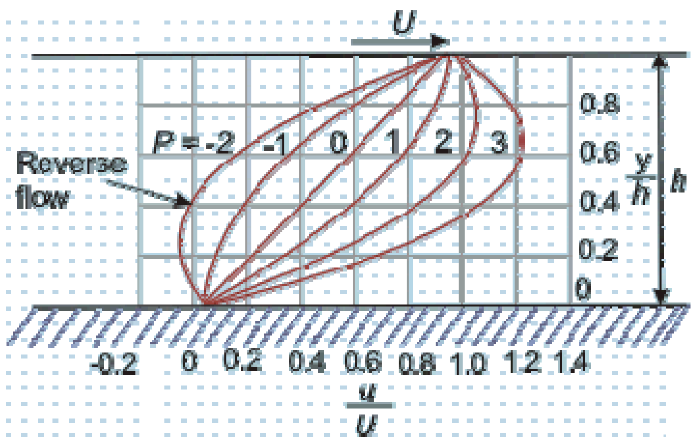


FIG 26.2b - Velocity distribution of the Couette flow

7.9 Hagen-Poisuelle's Flow:

Hagen Poiseuille Flow

- Consider **fully developed** laminar flow through a straight tube of circular cross-section as in Fig. 26.3. Rotational symmetry is considered to make the flow two-dimensional axisymmetric.
- Let us take z-axis as the axis of the tube along which all the fluid particles travel, i.e.

$$v_z \neq 0,\ v_r = 0, v_\theta = 0$$

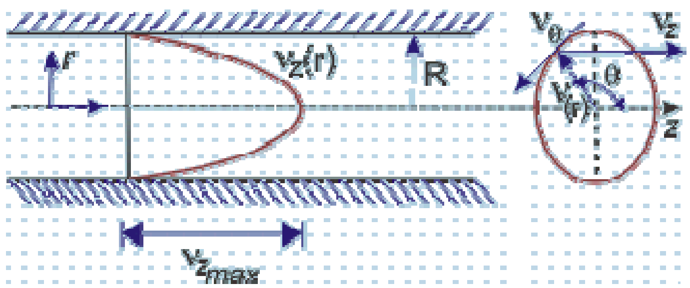


Fig 26.3 - Hagen-Poiseuille flow through a pipe

- Now, from continuity equation, we obtain

$$\cancel{\frac{\partial v_r}{\partial r}}^{0} + \cancel{\frac{v_r}{r}}^{0} + \frac{\partial v_z}{\partial z} = 0$$ [For rotational symmetry, $\frac{1}{r} \cdot \frac{\partial v_\theta}{\partial \theta} = 0$.]

which means $v_z = v_z(r, t)$

- Invoking $\left[v_r = 0, v_\theta = 0, \frac{\partial v_z}{\partial z} = 0, \text{ and } \frac{\partial}{\partial \theta}(\text{any quantity}) = 0 \right]$ in the

Navier-Stokes equations, we obtain

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial v_z}{\partial r} \right) \tag{26.3}$$ (in the z-direction)

- For **steady flow**, the governing equation becomes

$$\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial v_z}{\partial r} = \frac{1}{\mu} \frac{dp}{dz} \tag{26.4}$$

The ***boundary conditions*** are- **(i)** At r = 0, $v_z$is finite and **(ii)** r = R, $v_z = 0$ yields

- Equation (26.4) can be written as

$$r\frac{d^2 v_z}{dr^2}+\frac{dv_z}{dr}=\frac{1}{\mu}\cdot\frac{dp}{dz}r$$

or, $$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=\frac{1}{\mu}\cdot\frac{dp}{dz}r$$

or, $$r\frac{dv_z}{dr}=\frac{1}{2\mu}\cdot\frac{dp}{dz}r^2+A$$

or, $$\frac{dv_z}{dr}=\frac{1}{2\mu}\cdot\frac{dp}{dz}r+\frac{A}{r}$$

or, $$v_z=\frac{1}{4\mu}\cdot\frac{dp}{dz}r^2+A\ln r+B$$

- At r =0, $v_z$is finite which means A should be equal to zero and at r = R, v_z= 0 yields

$$B=-\frac{1}{4\mu}\cdot\frac{dp}{dz}\cdot R^2$$

$$v_z=\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left(1-\frac{r^2}{R^2}\right) \qquad (26.5)$$

- **This shows that the axial velocity profile in a fully developed laminar pipe flow is having parabolic variation along r.**
- At r = 0, as such, $v_z = v_{z_{max}}$

$$v_z = v_{z_{max}} = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right) \qquad (26.6a)$$

- The average velocity in the channel,

$$v_{z_{av}} = \frac{Q}{\pi R^2} = \frac{\int_0^R 2\pi r v_z(r)\,dr}{\pi R^2}$$

or, $$v_{z_{av}} = \frac{2\pi \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[\frac{R^2}{2} - \frac{R^4}{4R^2}\right]}{\pi R^2}$$

$$v_{z_{av}} = \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{1}{2} v_{z_{max}} \qquad (26.6b)$$

or $$v_{z_{max}} = 2 v_{z_{av}} \qquad (26.6c)$$

- Now, the discharge (Q) through a pipe is given by

$$Q = \pi R^2 v_{z_{av}} \qquad (26.7)$$

or, $$Q = \pi R^2 \frac{R^2}{8\mu}\left(\ \frac{dp}{dz}\right)$$ [From Eq. 26.6b]

or $$Q = -\frac{\pi D^4}{128\mu}\left(\frac{dp}{dz}\right) \qquad (26.8)$$

7.10 Hele Shaw Flow: Under Development

7.11 Flow through Co-rotating Cylinders:

Flow between Two Concentric Rotating Cylinders

- Another example which leads to an exact solution of Navier-Stokes equation is the flow between **two concentric rotating cylinders.**
- Consider flow in the annulus of two cylinders (Fig. 26.5), where r_1 and r_2 are the radii of inner and outer cylinders, respectively, and the cylinders move with different rotational speeds $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$ respectively

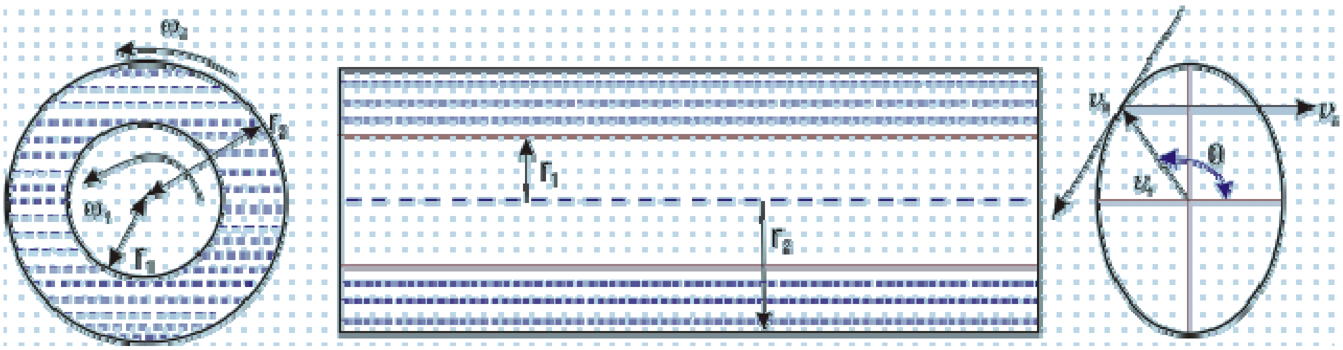


FIG 26.5 - Flow between two concentric rotating cylinders

- From the physics of the problem we know, , $v_z = 0, v_r = 0$.
- From the continuity Eq. and these two conditions, we obtain

$$\frac{\partial v_\theta}{\partial \theta} = 0$$

which means v_θ is not a function of θ. Assume z dimension to be large enough so that end effects can be neglected and $\frac{\partial}{\partial z}$ (any property) = 0.

- This implies $v_\theta = v_\theta(r)$. With these simplifications and assuming that " θ **symmetry**" holds good, Navier-Stokes equation reduces to

$$\rho \frac{v_\theta^2}{r} = \frac{dp}{dr} \tag{26.17}$$

and $$\frac{d^2 v_\theta}{dr^2} + \frac{1}{r} \cdot \frac{dv_\theta}{dr} - \frac{v_\theta}{r^2} = 0 \tag{26.18}$$

- Equation (26.17) signifies that the centrifugal force is supplied by the radial pressure, exerted by the wall of the enclosure on the fluid. In other words, it describes the **radial pressure distribution.**

  From Eq. (26.18), we get

$$\frac{d}{dr}\left[\frac{1}{r}\cdot\frac{d}{dr}(rv_\theta)\right]=0$$

$$\frac{d}{dr}(rv_\theta)Ar \quad \text{or} \quad v_\theta=\frac{Ar}{2}+\frac{B}{r} \qquad (26.19)$$

- For the azimuthal component of velocity, v_θ, the boundary conditions are: at $r=r_1, v_\theta=r_1\omega_1$ at $r=r_2, v_\theta=r_2\omega_2$.
- Application of these boundary conditions on Eq. (26.19) will produce

$$A=2\left[\omega_1-\frac{r_2^2}{(r_2^2-r_1^2)}(\omega_1-\omega_2)\right]$$

and

$$B=\frac{r_1^2 r_2^2}{(r_2^2-r_1^2)}(\omega_1-\omega_2)$$

- Finally, the velocity distribution is given by

$$v_\theta=\frac{1}{(r_2^2-r_1^2)}\left[\left(\omega_2 r_2^2-\omega_1 r_1^2\right)r+\frac{r_1^2 r_2^2(\omega_1-\omega_2)}{r}\right] \qquad (26.20)$$

Calculation of Stress and Torque Transmitted

Now, $\tau_{r\theta}=\mu\dot{\gamma}_{r\theta}$ is the general stress-strain relation.

or $$\tau_{r\theta}=\mu\left(\frac{\partial v_\theta}{\partial r}+\frac{1}{r}\cdot\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r}\right)$$

- In our case,

$$\tau_{r\theta} = \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

or $$\tau_{r\theta} = \mu r \frac{d}{dr}\left(\frac{v_\theta}{r}\right) \qquad (26.21)$$

- Equations (26.20) and (26.21) yields

$$\tau_{r\theta} = \frac{2\mu}{r_2^2 - r_1^2}(\omega_2 - \omega_1) r_1^2 r_2^2 \frac{1}{r^2} \qquad (26.22)$$

- Now,

$$\tau_{r\theta}\Big|_{r=r_1} = \frac{2\mu r_2^2(\omega_2 - \omega_1)}{r_2^2 - r_1^2}$$

and,

$$\tau_{r\theta}\Big|_{r=r_2} = \frac{2\mu r_1^2(\omega_2 - \omega_1)}{r_2^2 - r_1^2}$$

- For the case, when the inner cylinder is at rest and the outer cylinder rotates, the torque transmitted by the outer cylinder to the fluid is

$$T_2 = \frac{2\mu r_1^2 \omega_2}{r_2^2 - r_1^2} \cdot 2\pi r_2 l r_2$$

or $$T_2 = 4\pi\mu l \frac{r_2^2 r_1^2}{r_2^2 - r_1^2}\omega_2 \qquad (26.23)$$

where *l* is the length of the cylinder.

- The moment T_1, with which the fluid acts on the inner cylinder has the same magnitude. If the angular velocity of the external cylinder and the moment acting on the inner cylinder are measured, the coefficient of viscosity can be evaluated by making use of the Eq. (26.23).

7.12 Transition from Laminar to Turbulent Flows:

Mechanisms of Boundary Layer Transition

- One of the interesting problems in fluid mechanics is the physical mechanism of transition from laminar to turbulent flow. The problem evolves about the generation of both steady and unsteady vorticity near a body, its subsequent molecular diffusion, its kinematic and dynamic convection and redistribution downstream, and the resulting feedback on the velocity and pressure fields near the body. We can perhaps realise the complexity of the transition problem by examining the behaviour of a real flow past a cylinder.

  Figure 31.4 (a) shows the flow past a cylinder for a very low **Reynolds number** (~ 1). The **flow smoothly divides and reunites around the cylinder.**

- At a **Reynolds number of about 4**, the **flow (boundary layer) separates in the downstream** and the wake is formed by **two symmetric eddies** . The eddies remain steady and **symmetrical** but grow in size **up to a Reynolds number of about 40** as shown in Fig. 31.4(b).

- At a **Reynolds number above 40** , **oscillation in the wake** induces **asymmetry** and finally the wake starts **shedding vortices** into the stream. This situation is termed as **onset of periodicity** as shown in Fig. 31.4(c) and the wake keeps on undulating **up to a Reynolds number of 90** .

- At a **Reynolds number above 90** , the **eddies are shed alternately from a top and bottom** of the cylinder and the regular pattern of **alternately shed clockwise and counterclockwise vortices form *Von Karman vortex street*** as in Fig. 31.4(d).

- Periodicity is eventually induced in the flow field with the vortex-shedding phenomenon.

- The periodicity is characterised by the **frequency of vortex shedding** f

- **In non-dimensional form, the vortex shedding frequency is expressed as** fD/U_{ref} known as the **Strouhal number** named after V. Strouhal, a German physicist who experimented with wires singing in the wind. The Strouhal number shows a slight but continuous variation with Reynolds number around a value of 0.21. The boundary layer on the cylinder surface remains laminar and separation takes placeat about 81^0 from the forward stagnation point.

- **At about Re = 500** , multiple frequencies start showing up and the **wake tends to become Chaotic.**

- As the Reynolds number becomes higher, the boundary layer around the cylinder tends to become turbulent. The wake, of course, shows fully turbulent characters (Fig31.4 (e)).

- For larger Reynolds numbers, the boundary layer becomes turbulent. A turbulent boundary layer offers greater resistance to seperation than a laminar boundary layer. As a consequence the seperation point moves downstream and the seperation angle is delayed to 110^0 from the forward stagnation point (Fig 31.4 (f)).

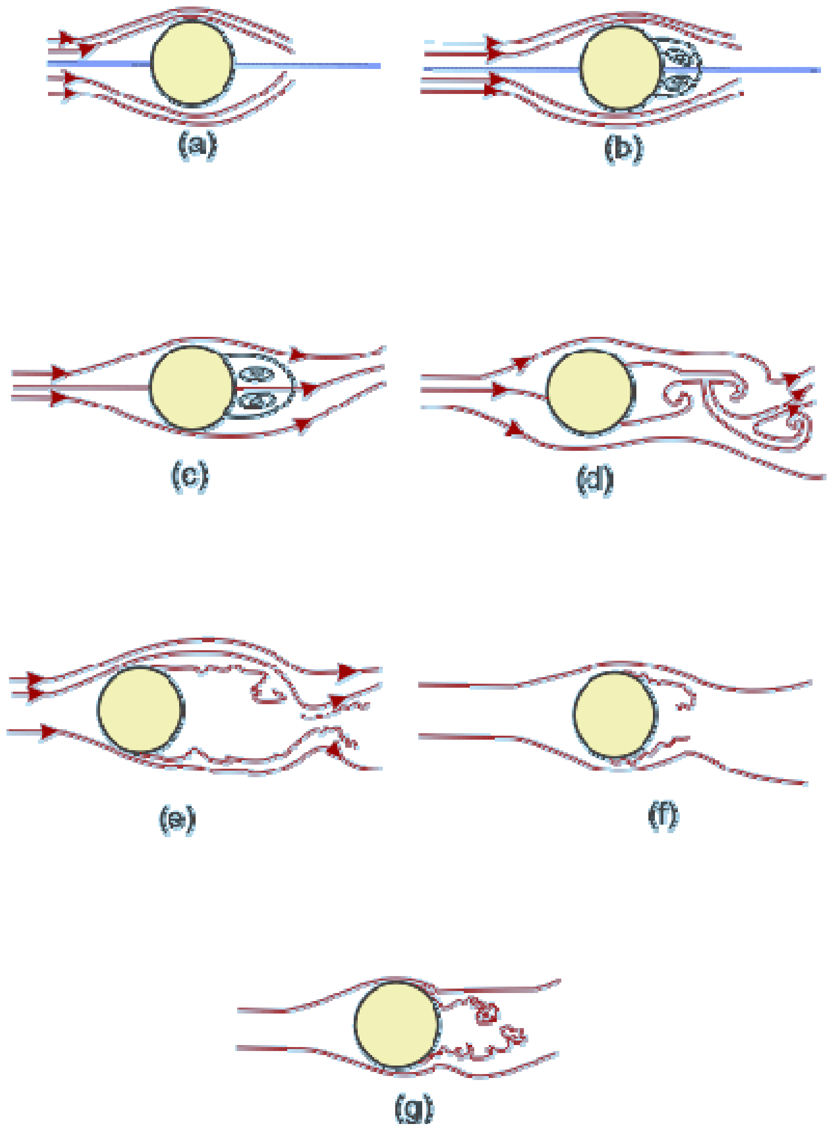


Fig. 31.4 Influence of Reynolds number on wake-zone aerodynamics

- Experimental flow visualizations past a circular cylinder are shown in *Figure 31.5 (a) and (b)*

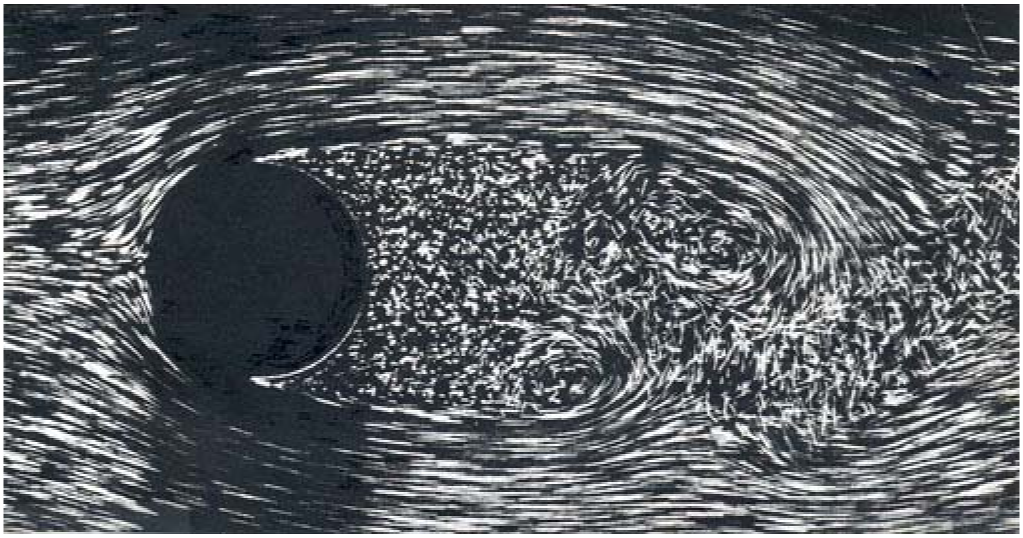


Fig 31.5 (a) Flow Past a Cylinder at Re=2000 [Photograph courtesy Werle and Gallon (ONERA)]

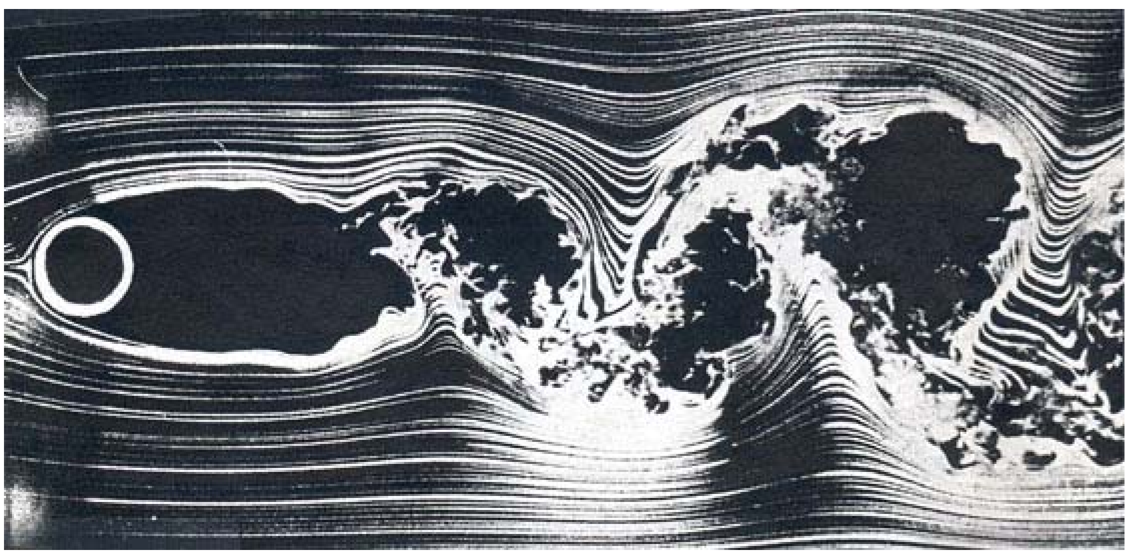


Fig 31.5 (b) Flow Past a Cylinder at Re=10000 [Photograph courtesy Thomas Corke and Hasan Najib (Illinois Institute of Technology, Chicago)]

- A very interesting sequence of events begins to develop when the Reynolds number is increased beyond 40, at which point the wake behind the cylinder becomes unstable. Photographs show that the wake develops a slow oscillation in which the velocity is periodic in time and downstream distance. The amplitude of the oscillation increases downstream. The oscillating wake rolls up into two staggered rows of vortices with opposite sense of rotation.

- Karman investigated the phenomenon and concluded that a nonstaggered row of vortices is unstable, and a staggered row is stable only if the ratio of lateral distance between the vortices to their longitudinal distance is 0.28. Because of the similarity of the wake with footprints in a street, the staggered row of vortices behind a blue body is called a **Karman Vortex Street** . The vortices move downstream at a speed smaller than the upstream velocity *U*.

- In the range 40 < Re < 80, the vortex street does not interact with the pair of attached vortices. As Re is increased beyond 80 the vortex street forms closer to the cylinder, and the attached eddies themselves begin to oscillate. Finally the attached eddies periodically break off alternately from the two sides of the cylinder.

- While an eddy on one side is shed, that on the other side forms, resulting in an unsteady flow near the cylinder. As vortices of opposite circulations are shed off alternately from the two sides, the circulation around the cylinder changes sign, resulting in an oscillating "lift" or lateral force. If the frequency of vortex shedding is close to the natural frequency of some mode of vibration of the cylinder body, then an appreciable lateral vibration culminates.

- Numerical flow visualizations for the flow past a circular cylinder can be observed in Fig 31.6 and 31.7

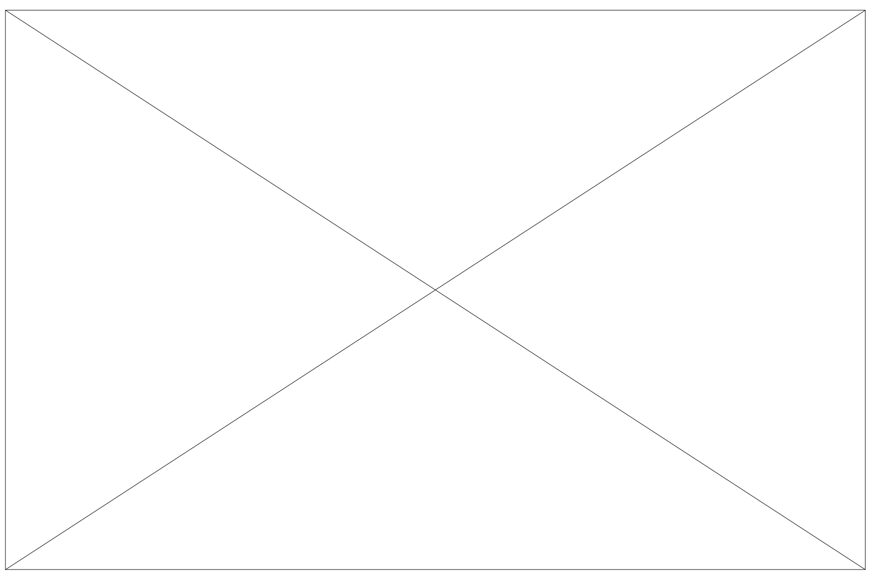


Fig 31.6 Numerical flow visualization (LES results) for a low reynolds number flow past a Circular Cylinder

[Animation by Dr.-Ing M. Breuer, LSTM, Univ Erlangen-Nuremberg]

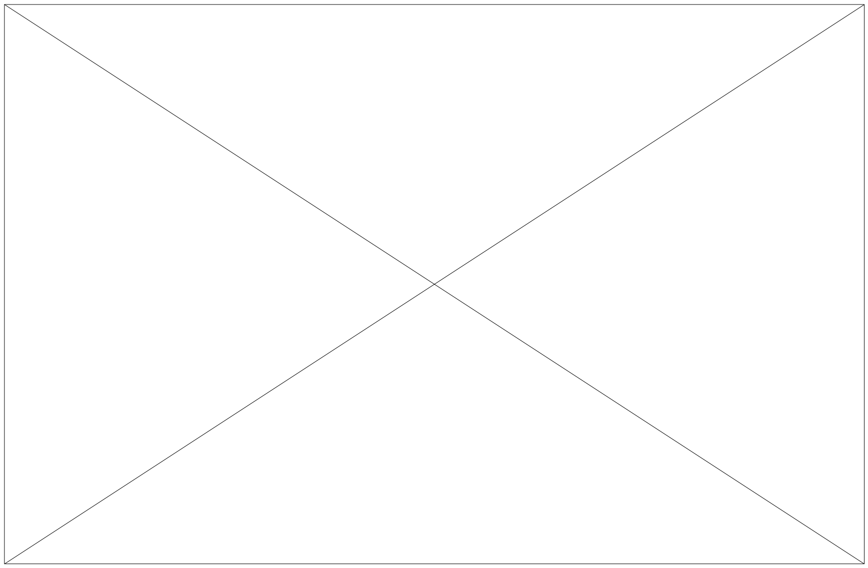


Fig 31.7 Numerical flow visualization (LES results) for a moderately high reynolds number flow past a Circular Cylinder
[Animation by Dr.-Ing M. Breuer, LSTM, Univ Erlangen-Nuremberg]

- An understanding of the transitional flow processes will help in practical problems either by improving procedures for predicting positions or for determining methods of advancing or retarding the transition position.

- The **critical value at which the transition occurs in pipe flow is** $Re_{cr} = 2300$. The actual value depends upon the disturbance in flow. Some experiments have shown the critical Reynolds number to reach as high as 10,000. The precise upper bound is not known, but the lower bound appears to be $Re_{cr} = 2300$. **Below this value, the flow remains laminar even when subjected to strong disturbances.**

- In the case of flow through a channel, $2300 \leq Re_{cr} \leq 2600$, the flow alternates randomly between laminar and partially turbulent. **Near the centerline, the flow is more laminar than turbulent, whereas near the wall, the flow is more turbulent than laminar.** For flow over a flat plate, turbulent regime is observed between Reynolds numbers $U_\infty x/\nu$ of 3.5×10^5 and 10^6.

Several Events Of Transition -

Transitional flow consists of several events as shown in *Fig. 31.8*. Let us consider the events one after another.

1. Region of instability of small wavy disturbances-

Consider a laminar flow over a flat plate aligned with the flow direction (*Fig. 31.8*).

- In the presence of an adverse pressure gradient, at a high Reynolds number (water velocity approximately 9-cm/sec), **two-dimensional waves appear.**
- **These waves are called *Tollmien-Schlichting* wave(** In 1929, Tollmien and Schlichting predicted that the waves would form and grow in the boundary layer).
- These waves can be made visible by a method known as tellurium method.

2. Three-dimensional waves and vortex formation-

- Disturbances in the free stream or **oscillations in the upstream boundary layer can generate wave growth,** which has a variation **in the span wise direction.**
- This leads an initially two-dimensional wave to **a three-dimensional form.**
- In many such transitional flows, periodicity is observed in the span wise direction.
- This is accompanied by the appearance of vortices whose axes lie in the direction of flow.

3. Peak-Valley development with streamwise vortices-

- As the three-dimensional wave propagates downstream, the **boundary layer flow develops into a complex stream wise vortex system.**
- Within this vortex system, **at some spanwise location, the velocities fluctuate violently** .
- These locations are **called peaks and the neighbouring locations of the peaks are valleys (***Fig. 31.9***).**

4. Vorticity concentration and shear layer development-

At the spanwise locations corresponding to the peak, the instantaneous streamwise velocity profiles demonstrate the following

- Often, an inflexion is observed on the velocity profile.
- The inflectional profile appears and disappears once after each cycle of the basic wave.

5. Breakdown-

The instantaneous velocity profiles produce high shear in the outer region of the boundary layer.

- The velocity fluctuations develop from the shear layer at a higher frequency than that of the basic wave.
- These velocity fluctuations have a strong ability to amplify any slight three-dimensionality, which is already present in the flow field.
- As a result, **a staggered vortex pattern evolves with the streamwise wavelength twice the wavelength of Tollmien-Schlichting wavelength .**
- The span wise wavelength of these structures is about one-half of the stream wise value.
- The high frequency fluctuations are referred as **hairpin eddies.**

This is known as breakdown.

6. Turbulent-spot development-

- The hairpin-eddies travel at a speed grater than that of the basic (primary) waves.
- As they travel downstream, eddies spread in the spanwise direction and towards the wall.
- The vortices begin a cascading breakdown into smaller vortices.
- In such a fluctuating state, intense local changes occur at random locations in the shear layer near the wall in the form of turbulent spots.
- Each spot grows almost linearly with the downstream distance.

The creation of spots is considered as the main event of transition .

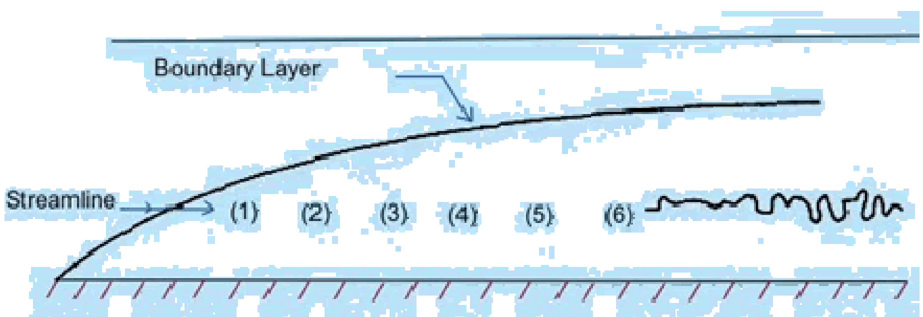


Fig. 31.8 Sequence of event involved in transition

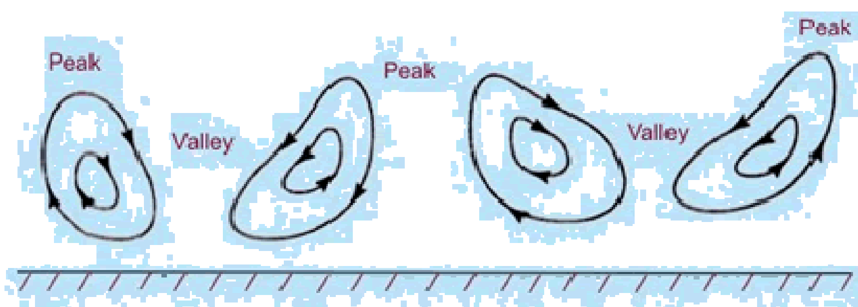


Fig. 31.9 Cross-stream view of the streamwise vortex system

Laminar-Turbulent Transition

- For a turbulent flow over a flat plate,

- The turbulent boundary layer continues to grow in thickness, with a small region below it called a **viscous sublayer**. In this sub layer, the flow is well behaved,just as the laminar boundary layer (Fig. 32.3)

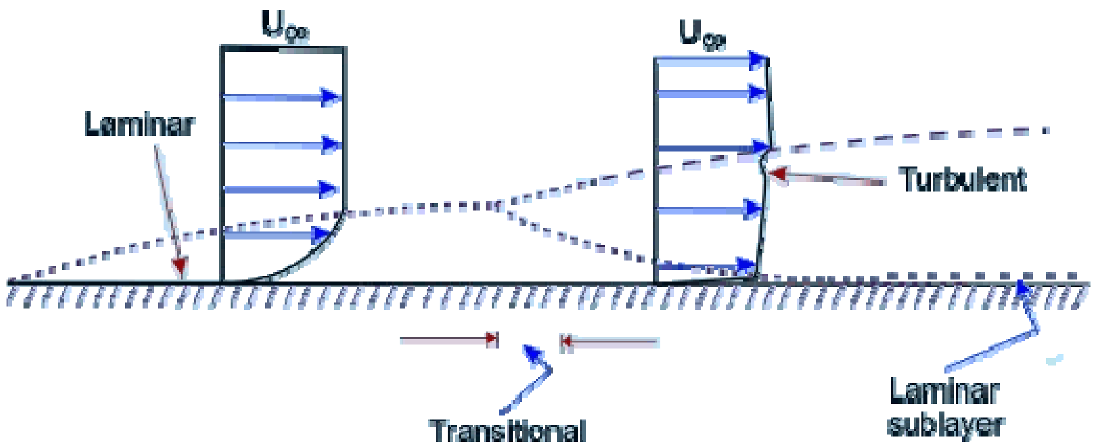
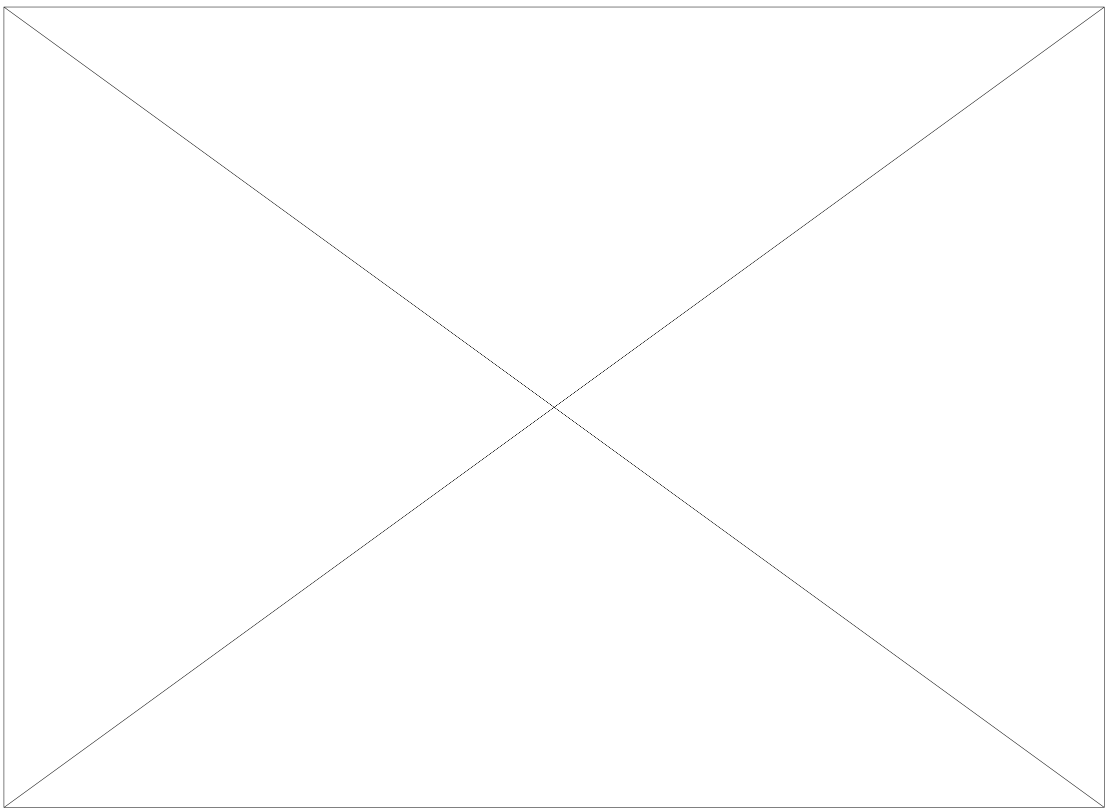


Fig. 32.3 Laminar - turbulent transition



Illustration

- Observe that at a certain axial location, the laminar boundary layer tends to become unstable. Physically this means that the disturbances in the flow grow in amplitude at this location.

Free stream turbulence, wall roughness and acoustic signals may be among the sources of such disturbances. **Transition to turbulent flow is thus initiated with the instability in laminar flow**

- The possibility of instability in boundary layer was felt by **Prandtl** as early as 1912.The theoretical analysis of **Tollmien and Schlichting** showed that unstable waves could exist if the **Reynolds number was 575**.

The Reynolds number was defined as

$$Re = U_\infty \delta^* / \nu$$

where U_∞ is the free stream velocity , δ^* is the displacement thickness and ν is the kinematic viscosity .

- **Taylor** developed an alternate theory, which assumed that the transition is caused by a momentary separation at the boundary layer associated with the free stream turbulence. In a pipe flow the initiation of turbulence is usually observed at **Reynolds numbers ($U_\infty D/\nu$)in the range of 2000 to 2700**.

The development starts with a laminar profile, undergoes a transition, changes over to turbulent profile and then stays turbulent thereafter (Fig. 32.4). The length of development is of the order of 25 to 40 diameters of the pipe.

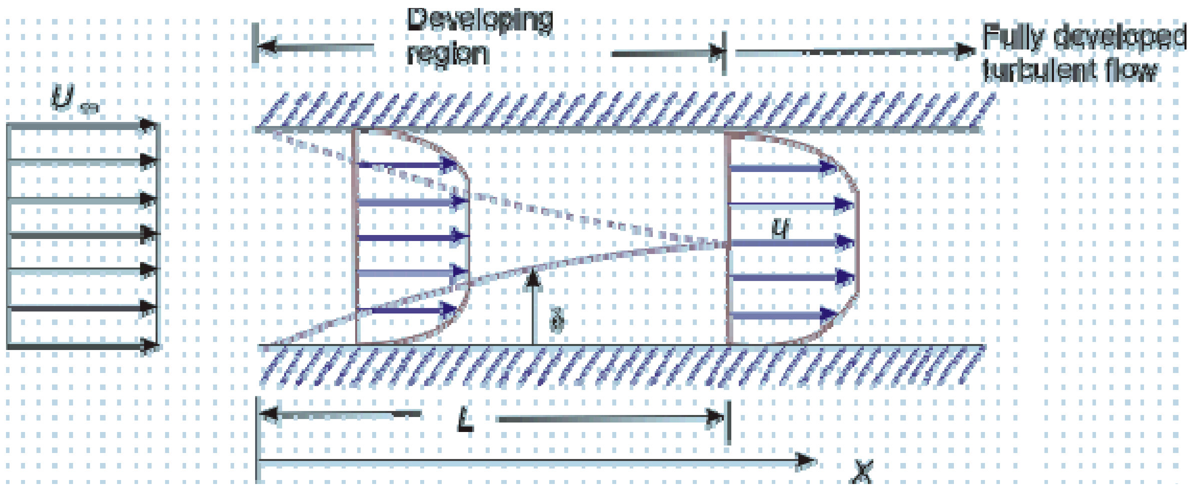


Fig. 32.4 Development of turbulent flow in a circular duct

7.13 Turbulent Flow in Circular Pipe:

Fully Developed Turbulent Flow In A Pipe For Moderate Reynolds Numbers

- The entry length of a turbulent flow is much shorter than that of a laminar flow, J. Nikuradse determined that a fully developed profile for turbulent flow can be observed after an entry length of 25 to 40 diameters. We shall focus to fully developed turbulent flow in this section.
- Considering a fully developed turbulent pipe flow (Fig. 34.3) we can write

$$2\pi R\tau_w = -\left(\frac{dp}{dx}\right)\pi R^2 \qquad (34.18)$$

or

$$\left(-\frac{dp}{dx} = \frac{2\tau_w}{R}\right) \qquad (34.19)$$

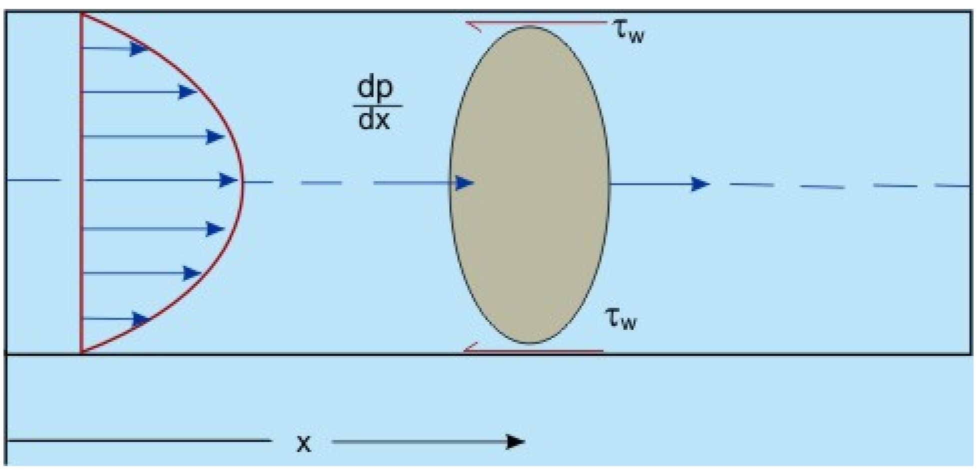


Fig. 34.3 Fully developed turbulent pipe flow

It can be said that in a fully developed flow, the pressure gradient balances the wall shear stress only and has a constant value at any x. However, the friction factor (Darcy friction factor) is defined in a fully developed flow as

$$-\left(\frac{dp}{dx}\right) = \frac{f\rho U_{av}^2}{2D} \qquad (34.20)$$

Comparing Eq.(34.19) with Eq.(34.20), we can write

$$\tau_w = \frac{f}{8}\rho U_{av}^2 \qquad (34.21)$$

H. Blasius conducted a critical survey of available experimental results and established the empirical correlation for the above equation as

$$f = 0.3164\,\mathrm{Re}^{-0.25} \text{ where } \mathrm{Re} = \rho U_{av} D/\mu \qquad (34.22)$$

- It is found that the Blasius's formula is valid in the range of Reynolds number of Re $\leq 10^5$. At the time when Blasius compiled the experimental data, results for higher Reynolds numbers were not available. However, later on, J. Nikuradse carried out experiments with the laws of friction in a very wide range of Reynolds numbers, $4 \times 10^3 \leq \mathrm{Re} \leq 3.2 \times 10^6$. The velocity profile in this range follows:

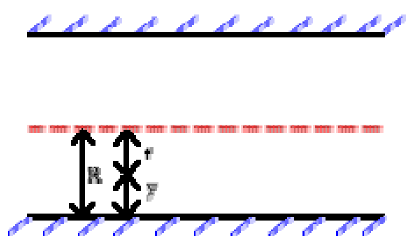
$$\frac{u}{\bar{u}} = \left[\frac{y}{R}\right]^{1/n} \qquad (34.23)$$

where $\bar{u}$ is the time mean velocity at the pipe centre and y is the distance from the wall . The exponent *n* varies slightly with Reynolds number. In the range of Re ~ 10^5, *n* is

- The ratio of $\bar{u}$ and U_{av} for the aforesaid profile is found out by considering the volume flow rate *Q* as

$$Q = \pi R^2 U_{av} = \int_0^R 2\pi r u\, dr$$

$$r = R - y$$



From equation (34.23)

$$\pi R^2 U_{av} = 2\pi \bar{u} \int_R^0 (R - y)(y/R)^{1/n}(-dy)$$

or

$$\pi R^2 U_{av} = 2\pi \bar{u}\left[\frac{n}{n+1}\left(R^{\frac{n-1}{n}} y^{\frac{n+1}{n}}\right) - \frac{n}{2n+1}\left(y^{\frac{2n+1}{n}} R^{-\frac{1}{n}}\right)\right]_0^R$$

or

$$\pi R^2 U_{av} = 2\pi \bar{u}\left[R^2 \frac{n}{n+1} - \frac{n}{2n+1} R^2\right]$$

or

$$\pi R^2 U_{av} = 2\pi R^2 \bar{u}\left[\frac{n^2}{(n+1)(2n+1)}\right]$$

or

$$\frac{U_{av}}{\bar{u}} = \frac{2n^2}{(n+1)(2n+1)} \qquad (34.24a)$$

- Now, for different values of n (for different Reynolds numbers) we shall obtain different values of $U_{av}/\bar{u}$ from Eq.(34.24a). On substitution of Blasius resistance formula (34.22) in Eq.(34.21), the following expression for the shear stress at the wall can be obtained.

$$\tau_w = \frac{0.3164}{8}\mathrm{Re}^{-0.25}\rho U_{av}^2$$

putting $\mathrm{Re} = \rho U_{av} 2R/\mu$

and where $\nu = \mu/\rho$

$$\tau_w = 0.03955\rho U_{av}^2\left(\frac{\nu}{2RU_{av}}\right)^{1/4}$$

or

$$\tau_w = 0.03325\rho U_{av}^{7/4}\left(\frac{\nu}{R}\right)^{1/4}$$

or

$$\tau_w = 0.03325\rho\left(\frac{U_\infty}{\bar{u}}\right)^{7/4}(\bar{u})^{7/4}\left(\frac{\nu}{R}\right)^{1/4}$$

- For n=7, $U_\infty/\bar{u}$ becomes equal to 0.8. substituting $U_\infty/\bar{u} = 0.8$ in the above equation, we get

$$\tau_w = 0.03325\rho(0.8)^{1/4}(\bar{u})^{7/4}(\nu/R)^{1/4}$$

Finally it produces

$$\tau_w = 0.0225\rho(\bar{u})^{7/4}(\nu/R)^{1/4} \tag{34.24b}$$

or

$$u_\tau^2\rho = 0.0225\rho(\bar{u})^{7/4}\left(\frac{\nu}{R}\right)^{1/4}$$

where u_τ is friction velocity. However, u_τ^2 may be spitted into $u_\tau^{1/4}$ and $u_\tau^{7/4}$ and we obtain

$$\left(\frac{\bar{u}}{u_\tau}\right)^{7/4} = 44.44\left(\frac{u_\tau R}{\nu}\right)^{1/4}$$

or

$$\frac{\bar{u}}{u_\tau} = 8.74\left(\frac{u_\tau R}{\nu}\right)^{1/7} \tag{34.25a}$$

- Now **we can assume that the above equation is not only valid at the pipe axis (y = R) but also at any distance** from the wall y and a **general form is proposed as**

$$\frac{\bar{u}}{u_\tau} = 8.74\left(\frac{y u_\tau}{\nu}\right)^{1/7} \tag{34.25b}$$

- Concluding Remarks :

1. It can be said that (1/7)th power velocity distribution law (24.38b) can be derived from Blasius's resistance formula (34.22) .
2. Equation (34.24b) gives the shear stress relationship in pipe flow at a moderate Reynolds number, i.e $Re \leq 10^5$. Unlike very high Reynolds number flow, here laminar effect

cannot be neglected and the laminar sub layer brings about remarkable influence on the outer zones.

3. The friction factor for pipe flows, f , defined by Eq. (34.22) is valid for a specific range of Reynolds number and for a particular surface condition.

Chapter 8
Introduction to Incompressible Boundary Layer

8.1 Boundary Layer Concept:

Introduction

- The **boundary layer** of a flowing fluid is **the thin layer close to the wall**
- In a flow field, **viscous stresses are very prominent within this layer**.
- Although the layer is thin, it is very important to know the details of flow within it.
- The **main-flow velocity** within this layer **tends to zero** while approaching the wall **(no-slip condition)**.
- Also the gradient of this velocity component in a direction normal to the surface is large as compared to the gradient in the streamwise direction.

8.2 Boundary Layer Properties:

Boundary Layer Equations

- In 1904, **Ludwig Prandtl,** the well known German scientist, introduced the concept of boundary layer and **derived the equations for boundary layer flow** by correct reduction of Navier-Stokes equations.

- He hypothesized that **for fluids having relatively small viscosity**, **the effect of internal friction in the fluid is significant only in a narrow region surrounding solid boundaries or bodies over which the fluid flows.**

- Thus, close to the body is the boundary layer where **shear stresses exert an increasingly larger effect** on the fluid **as one moves from free stream towards the solid boundary.**

- However, **outside the boundary layer where the effect of the shear stresses on the flow is small compared to values inside the boundary layer (since the velocity gradient** $\partial u/\partial y$ **is negligible)**,---------

1. the fluid particles experience **no vorticity** and therefore**,**
2. the flow is similar to a **potential flow.**

- Hence, the ***surface*** **at the boundary layer interface** is a rather fictitious one, that **divides rotational and irrotational flow.** Fig 28.1 shows Prandtl's model regarding boundary layer flow.

- Hence with the exception of the immediate vicinity of the surface, the flow is frictionless (inviscid) and the velocity is *U* (the potential velocity).

- In the region, very near to the surface (in the thin layer), there is friction in the flow which signifies that the fluid is retarded until it adheres to the surface (**no-slip condition).**

- The transition of the mainstream velocity from zero at the surface (with respect to the surface) to full magnitude takes place across the boundary layer.

About the boundary layer

- Boundary layer **thickness is** δ which is a **function of** the coordinate direction ***x*** .
- The thickness is considered to be **very small compared to the characteristic length *L*** of the domain.
- In the normal direction, **within this thin layer**, the gradient $\partial u/\partial y$ **is very large compared to** the gradient in the flow direction $\partial u/\partial x$**.**

Now we take up the Navier-Stokes equations for : steady, two dimensional, laminar, incompressible flows.

Considering the Navier-Stokes equations together with the equation of continuity, the following dimensional form is obtained.

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \quad (28.1)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}--\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \quad (28.2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (28.3)$$

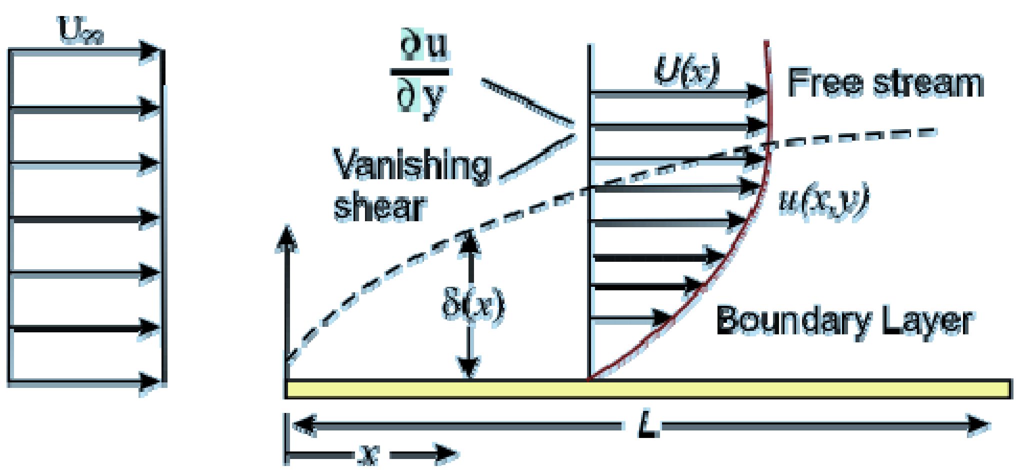


Fig 28.1 Boundary layer and Free Stream for Flow Over a flat plate

- u - velocity component along x direction.
- v - velocity component along y direction
- p - static pressure
- ρ - density.
- μ - dynamic viscosity of the fluid

- The equations are now non-dimensionalised.
- The l**ength and the velocity scales** are chosen as ***L*** **and** U_∞**respectively.**
- The non-dimensional variables are:

$$u^* = \frac{u}{U_\infty}, v^* = \frac{v}{U_\infty}, p^* = \frac{p}{\rho U_\infty^2}$$

$$x^* = \frac{x}{L}, y^* = \frac{y}{L}$$

where $U_\infty$is the dimensional free stream velocity and the pressure is non-dimensionalised by twice the dynamic pressure $p_d = (1/2)\rho U_\infty^2$.

Using these non-dimensional variables, the Eqs (28.1) to (28.3) become

$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right]$	(28.4)
$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right]$	(28.5)
$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$	(28.6)

where the Reynolds number,

$$\mathrm{Re} = \frac{\rho U_\infty L}{\mu}$$

8.3 Derivation of Prandtl Bounday Layer Equation:

Order of Magnitude Analysis

- Let us examine what happens to the *u* velocity as we go across the boundary layer. At the **wall** the *u* velocity is **zero** [with respect to the wall and absolute zero for a stationary wall (which is normally implied if not stated otherwise)]. The value of *u* on the **inviscid side**, that is on the free stream side beyond the boundary layer is ***U.*** For the case of external flow over a flat plate, this ***U*** is equal to U_∞.
- Based on the above, we can identify the following scales for the boundary layer variables:

Variable	*Dimensional scale*	*Non-dimensional scale*
u	U_∞	1
x	L	1
y	δ	$\varepsilon = \delta / L$

- The **symbol** ε **describes a value much smaller than 1.**
- Now we analyse equations 28.4 - 28.6, and look at the order of magnitude of each individual term

Eq 28.6 - the continuity equation

One **general rule** of incompressible fluid mechanics is that **we are not allowed to drop any term from the continuity equation**.

- From the scales of boundary layer variables, the derivative $\partial u^* / \partial x^*$ is of the order 1.
- The second term in the continuity equation $\partial v^* / \partial y^*$ should also be of the order 1.The reason being v^* has to be of the order ε because y^* becomes $\varepsilon (= \delta / L)$ at its maximum.

Eq 28.4 - x direction momentum equation

- Inertia terms are of the order 1.
- $\partial^2 u / \partial x^2$ is of the order 1
- $\partial^2 u / \partial y^2$ is of the order $(1 / \varepsilon^2)$.

However after multiplication with 1/Re, the sum of the two second order derivatives should produce at least one term which is of the same order of magnitude as the inertia terms. This is possible only if the Reynolds number (Re) is of the order of $(1/\varepsilon^2)$.

- It follows from that $-\partial p^*/\partial x^*$ will not exceed the order of 1 so as to be in balance with the remaining term.
- Finally, Eqs (28.4), (28.5) and (28.6) can be rewritten as

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\left[\frac{\partial^2 u^*}{\partial x^{*2}}-\frac{\partial^2 u^*}{\partial y^{*2}}\right] \qquad (28.4)$$

$$(1)\frac{(1)}{(1)} \qquad (\varepsilon)\frac{(\ )}{(\varepsilon)} \; -(1) \qquad (\varepsilon^2)\left[\frac{(1)}{(1)} \;\; \frac{1}{(\varepsilon^2)}\right]$$

$$u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\left[\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right] \qquad (28.5)$$

$$(1)\frac{(\varepsilon)}{(1)} \qquad (\varepsilon)\frac{(\varepsilon)}{(\varepsilon)} = (?) \qquad (\varepsilon^2)\left[\frac{(\varepsilon)}{(1)}+\frac{\varepsilon}{(\varepsilon^2)}\right]$$

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \qquad (28.6)$$

$$\frac{(1)}{(1)} \qquad \frac{(\varepsilon)}{(\varepsilon)}$$

As a consequence of the order of magnitude analysis, $\partial^2 u^*/\partial x^{*2}$ can be dropped from the *x* direction momentum equation, because on multiplication with $1/Re$ it assumes the smallest order of magnitude.

Eq 28.5 - *y* direction momentum equation.

- All the terms of this equation are of a smaller magnitude than those of Eq. (28.4).

- This equation can only be balanced if $\partial p^*/\partial y^*$ is of the same order of magnitude as other terms.
- Thus they momentum equation reduces to

$$\frac{\partial p^*}{\partial y^*} = O(\varepsilon) \tag{28.7}$$

- This means that the **pressure across the boundary layer does not change**. The **pressure is impressed on the boundary layer**, and its value is determined by hydrodynamic considerations.
- This also implies that the **pressure *p* is only a function of *x***. The pressure forces on a body are solely **determined by the inviscid flow outside the boundary layer.**
- The application of Eq. (28.4) at the outer edge of boundary layer gives

$$u^* \frac{\partial u^*}{\partial x^*} = -\frac{\partial p^*}{\partial x^*} \tag{28.8a}$$

In dimensional form, this can be written as

$$U \frac{dU}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{28.8b}$$

On integrating Eq (28.8b) the well known Bernoulli's equation is obtained

$$p + \frac{1}{2}\rho U^2 = \textit{a constant} \tag{28.9}$$

- Finally, it can be said that by the order of magnitude analysis, the Navier-Stokes equations are simplified into equations given below.

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}} \frac{\partial^2 u^*}{\partial y^{*2}} \tag{28.10}$$

$$\frac{\partial p^*}{\partial y^*} = 0 \qquad (28.11)$$

-

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \qquad (28.12)$$

-
- These are known as Prandtl's boundary-layer equations.

The available boundary conditions are:

Solid surface $at\ y^* = 0, u^* = 0 = v^*$

or $at\ y = 0, u = 0 = v$ (28.13)

Outer edge of boundary-layer

$$at\ y^* = (x) = \frac{\delta}{L}, u^* = 1$$

or $at\ y = \delta, u = U(x)$ (28.14)

- The unknown pressure *p* in the x-momentum equation can be determined from Bernoulli's Eq. (28.9), if the inviscid velocity distribution *U(x)* is also known.

We solve the Prandtl boundary layer equations for $u^*(x,y)$ and $v^*(x,y)$ with *U* obtained from the outer inviscid flow analysis. The equations are solved by commencing at the leading edge of the body and moving downstream to the desired location

- it allows the no-slip boundary condition to be satisfied which constitutes a significant improvement over the potential flow analysis while solving real fluid flow problems.
- The **Prandtl boundary layer equations** are thus a simplification of the Navier-Stokes equations.

8.4 Blasius Solution:

Blasius Flow Over A Flat Plate

- The classical problem considered by H. Blasius was
    1. Two-dimensional, steady, incompressible flow over a flat plate at zero angle of incidence with respect to the uniform stream of velocity U_∞.
    2. The fluid extends to infinity in all directions from the plate.

The physical problem is already illustrated in Fig. 28.1

- Blasius wanted to determine
  (a) the velocity field solely within the boundary layer,
  (b) the boundary layer thickness (δ),
  (c) the shear stress distribution on the plate, and
  (d) the drag force on the plate.
- The Prandtl boundary layer equations in the case under consideration are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \qquad (28.15)$$

$$\nu=\mu/\rho$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

The boundary conditions are

$$at\ y=0,\ u=v=0 \qquad (28.16)$$

$$at\quad y=\infty,\quad u=U_\infty$$

- Note that the substitution of the term $-\frac{1}{\rho}\frac{dp}{dx}$ in the original boundary layer momentum equation in terms of the free stream velocity produces $U_\infty\frac{dU_\infty}{dx}$ which is equal to zero.
- Hence the governing Eq. (28.15) does not contain any pressure-gradient term.

- However, the characteristic parameters of this problem are U_∞, ν, x, y that is, $u = u(U_\infty, \nu, x, y)$
- This relation has five variables U_∞, ν, x, y.
- It involves two dimensions, length and time.
- Thus it can be reduced to a dimensionless relation in terms of (5-2) =3 quantities (**Buckingham Pi Theorem**)
- Thus a similarity variables can be used to find the solution
- Such flow fields are called self-similar flow field .

Law of Similarity for Boundary Layer Flows

- It states that the *u* component of velocity with two velocity profiles of *u(x,y)* at different *x* locations differ only by scale factors in *u* and *y* .
- Therefore, the velocity profiles *u(x,y)* at all values of *x* can be made congruent if they are plotted in coordinates which have been made dimensionless with reference to the scale factors.
- The local free stream velocity *U(x)* at section *x* is an obvious scale factor for *u*, because the dimensionless *u(x)* varies between zero and unity with *y* at all sections.
- The scale factor for *y* , denoted by *g(x)* , is proportional to the local boundary layer thickness so that *y* itself varies between zero and unity.
- Velocity at two arbitrary *x* locations, namely x_1 and x_2 should satisfy the equation

$$\frac{u[x_1, (y/g(x_1))]}{U(x_1)} = \frac{u[x_2, (y/g(x_2))]}{U(x_2)} \qquad (28.17)$$

- Now, for Blasius flow, it is possible to identify g(x) with the boundary layers thickness δ we know

$$\varepsilon = \frac{\delta}{L} \sim \frac{1}{\sqrt{\mathrm{Re}_L}}$$

Thus in terms of x we get

$$\frac{\delta}{x} \sim \frac{1}{\sqrt{\frac{U_\infty x}{\nu}}}$$

$$\delta \sim \sqrt{\frac{\nu x}{U_\infty}}$$

i.e.,

$$\frac{u}{U_\infty} = F\left(\frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}\right) = F(\eta) \qquad (28.18)$$

where $\eta \sim \frac{y}{\delta}$ and $\delta \sim \sqrt{\frac{\nu x}{U_\infty}}$
or more precisely,

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}} \qquad (28.19)$$

$$y = \eta\sqrt{\frac{\nu x}{U_\infty}}$$

$$dy = \sqrt{\frac{\nu x}{U_\infty}}d\eta$$

The stream function can now be obtained in terms of the velocity components as

$$\psi = \int u dy = \int U_\infty F(\eta)\sqrt{\frac{\nu x}{U_\infty}}d\eta = \sqrt{U_\infty \nu x}\int F(\eta)d\eta$$

or

$$\psi = \sqrt{U_\infty \nu x} f(\eta) + D \qquad (28.20)$$

where D is a constant. Also $\int F(\eta)d\eta = f(\eta)$ and the constant of integration is zero if the stream function at the solid surface is set equal to zero.

Now, the velocity components and their derivatives are:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}.\frac{\partial \eta}{\partial y} = U_\infty f'(\eta) \tag{28.21a}$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{U_\infty v}\left[\frac{1}{2}.\frac{1}{\sqrt{x}}f(\eta) + \sqrt{x}f'(\eta)\left\{-\frac{1}{2}\frac{y}{\sqrt{vx/U_\infty}}\frac{1}{x}\right\}\right]$$

or

$$v = \frac{1}{2}\sqrt{\frac{vU_\infty}{x}}\left[\eta f'(\eta) - f(\eta)\right] \tag{28.21b}$$

$$\frac{\partial u}{\partial x} = U_\infty f''(\eta)\frac{\partial \eta}{\partial x} = U_\infty f''(\eta)\left[-\frac{1}{2}.\frac{y}{\sqrt{vx/U_\infty}}\frac{1}{x}\right]$$

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2}.\frac{\eta}{x}f''(\eta) \tag{28.21c}$$

$$\frac{\partial u}{\partial y} = U_\infty f''(\eta)\frac{\partial \eta}{\partial y} = U_\infty f''(\eta)\left[\frac{1}{\sqrt{vx/U_\infty}}\right]$$

$$\frac{\partial u}{\partial y} = U_\infty\sqrt{\frac{U_\infty}{vx}}f''(\eta) \tag{28.21d}$$

$$\frac{\partial^2 u}{\partial y^2} = U_\infty\sqrt{\frac{U_\infty}{vx}}f'''(\eta)\left\{\frac{1}{\sqrt{vx/U_\infty}}\right\}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{vx}f'''(\eta) \tag{28.21e}$$

- Substituting (28.2) into (28.15), we have

$$-\frac{U_\infty^2}{2}\frac{\eta}{x}f'(\eta)f''(\eta)+\frac{U_\infty^2}{2x}\left[\eta f'(\eta)-f(\eta)\right]f''(\eta)=\frac{U_\infty^2}{x}f'''(\eta)$$

$$-\frac{1}{2}\frac{U_\infty^2}{x}f(\eta)f''(\eta)=\frac{U_\infty^2}{x}f'''(\eta)$$

or,

$$2f'''(\eta)+f(\eta)f''(\eta)=0 \qquad (28.22)$$

where

$$f(\eta)=\int F(\eta)d\eta+C$$

$$=\int\frac{u}{U_\infty}d\eta+C$$

and

$$\eta=\frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}$$

This is known as Blasius Equation .

8.5 Karmans Integral Equation:

Momentum-Integral Equations For The Boundary Layer

- To employ boundary layer concepts in real engineering designs, we need approximate methods that would quickly lead to an answer even if the accuracy is somewhat less.
- **Karman and Pohlhausen** devised a simplified method by **satisfying only the boundary conditions of the boundary layer flow** rather than satisfying Prandtl's differential equations for each and every particle within the boundary layer. We shall discuss this method herein.
- Consider the case of steady, two-dimensional and incompressible flow, i.e. we shall refer to Eqs (28.10) to (28.14). Upon integrating the dimensional form of Eq. (28.10) with respect to $y = 0$ (wall) to $y = \delta$ (where δ signifies the interface of the free stream and the boundary layer), we obtain

$$\int_0^\delta \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) dy = \int_0^\delta \left(-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \right) dy$$

or,
$$\int_0^\delta u\frac{\partial u}{\partial x}dy + \int_0^\delta v\frac{\partial u}{\partial y}dy = \int_0^\delta -\frac{1}{\rho}\frac{\partial p}{\partial x}dy + \int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}dy \qquad (29.10)$$

- The second term of the left hand side can be expanded as

$$\int_0^\delta v\frac{\partial u}{\partial y}dy = [vu]_0^\delta \quad \int_0^\delta u\frac{\partial v}{\partial y}dy$$

or,
$$\int_0^\delta v\frac{\partial u}{\partial y}dy = U_\infty v_\delta + \int_0^\delta u\frac{\partial u}{\partial x}dy \left(\text{since } \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \right)$$ by continuity equation

or,
$$\int_0^\delta v\frac{\partial u}{\partial y}dy = -U_\infty \int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta u\frac{\partial u}{\partial x}dy \qquad (29.11)$$

- Substituting Eq. (29.11) in Eq. (29.10) we obtain

$$\int_0^{\delta} 2u\frac{\partial u}{\partial x}dy - U_\infty\int_0^{\delta}\frac{\partial u}{\partial x}dy = -\int_0^{\delta}\frac{1}{\rho}\frac{\partial p}{\partial x}dy - \nu\left.\frac{\partial u}{\partial y}\right|_{y=0} \quad (29.12)$$

- Substituting the relation between $\frac{\partial p}{\partial x}$ and the free stream velocity U_∞ for the inviscid zone in Eq. (29.12) we get

$$\int_0^{\delta} 2u\frac{\partial u}{\partial x}dy - U_\infty\int_0^{\delta}\frac{\partial u}{\partial x}dy - \int_0^{\delta}U_\infty\frac{dU_\infty}{dx}dy = -\left(\frac{\mu\left.\frac{\partial u}{\partial y}\right|_{y=0}}{\rho}\right)$$

$$\int_0^{\delta}\left(2u\frac{\partial u}{\partial x} - U_\infty\frac{\partial u}{\partial x} - U_\infty\frac{dU_\infty}{dx}\right)dy = -\frac{\tau_w}{\rho}$$

which is reduced to

$$\int_0^{\delta}\frac{\partial}{\partial x}\left[u(U_\infty - u)\right]dy + \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)dy = \frac{\tau_w}{\rho}$$

- Since the integrals vanish outside the boundary layer, we are allowed to increase the integration limit to infinity (i.e $\delta = \infty$.)

$$\int_0^{\delta}\frac{\partial}{\partial x}\left[u(U_\infty - u)\right]dy + \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)dy = \frac{\tau_w}{\rho}$$

or, $$\frac{d}{dx}\int_0^{\delta}\left[u(U_\infty - u)\right]dy + \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)dy = \frac{\tau_w}{\rho} \quad (29.13)$$

- Substituting Eq. (29.6) and (29.7) in Eq. (29.13) we obtain

$$\frac{d}{dx}\left[U_\infty^2 \delta^{**}\right] + \delta^* U_\infty \frac{dU_\infty}{dx} = \frac{\tau_w}{\rho} \qquad (29.14)$$

where $\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$ is the displacement thickness

$\delta^{**} = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$ is momentum thickness

Equation (29.14) is known as momentum integral equation for two dimensional incompressible laminar boundary layer. The same remains valid for turbulent boundary layers as well.

Needless to say, **the wall shear stress (τ_w) will be different for laminar and turbulent flows.**

- The term $U_\infty \frac{dU_\infty}{dx}$ signifies space-wise acceleration of the free stream. Existence of this term means that free stream pressure gradient is present in the flow direction.
- For example, we get finite value of $U_\infty \frac{dU_\infty}{dx}$ outside the boundary layer in the entrance region of a pipe or a channel. For external flows, the existence of $U_\infty \frac{dU_\infty}{dx}$ depends on the shape of the body.
- During the flow over a flat plate, $U_\infty \frac{dU_\infty}{dx} = 0$ and the momentum integral equation is reduced to

$$\frac{d}{dx}\left[U_\infty^2 \delta^{**}\right] = \frac{\tau_w}{\rho} \qquad (29.15)$$

Karman-Pohlhausen Approximate Method For Solution Of Momentum Integral Equation Over A Flat Plate

- The basic equation for this method is obtained by integrating the *x* direction momentum equation (boundary layer momentum equation) with respect to *y* from the wall (at y = 0) to a distance $\delta(x)$ which is assumed to be outside the boundary layer. Using this notation, we can rewrite the Karman momentum integral equation as

$$U_\infty^2 \frac{d\delta^{**}}{dx} + \left(2\delta^{**} + \delta^{*}\right) U_\infty \frac{dU_\infty}{dx} = \frac{\tau_w}{\rho} \tag{30.1}$$

- The effect of pressure gradient is described by the second term on the left hand side. For pressure gradient surfaces in external flow or for the developing sections in internal flow, this term contributes to the pressure gradient.
- We assume **a velocity profile which is a polynomial of** $\eta = y/\delta$. η being a form of **similarity variable** , implies that **with the growth of boundary layer as distance *x* varies from the leading edge, the velocity profile** (u/U_∞) **remains geometrically similar.**
- We choose a velocity profile in the form

$$\frac{u}{U_\infty} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 \tag{30.2}$$

- In order to determine the constants a_0, a_1, a_2 and a_3 we shall prescribe the following boundary conditions

$$at\ y = 0, u = 0 \quad or \quad at\ \eta = 0, \frac{u}{U_\infty} = 0 \tag{30.3a}$$

$$at\ y = 0, \frac{\partial^2 u}{\partial y^2} = 0 \quad or \quad at\ \eta = 0, \frac{\partial^2}{\partial \eta^2}(u/U_\infty) = 0 \tag{30.3b}$$

- at

$$at\ y = \delta, u = U_\infty \quad or \quad at\ \eta = 1, \frac{u}{U_\infty} = 1 \tag{30.3c}$$

- at

$$at\ y = \delta, \frac{\partial u}{\partial y} = 0 \quad or \quad at\ \eta = 1, \frac{\partial(u/U_\infty)}{\partial \eta} = 0 \tag{30.3d}$$

- These requirements will yield $a_0 = 0, a_2 = 0, a_1 + a_3 = 1$ and $a_1 + 3a_3 = 0$ respectively Finally, we obtain the following values for the coefficients in Eq. (30.2),

$a_0 = 0, a_1 = 3/2, a_2 = 0$ and $+a_3 = -1/2$ and the velocity profile becomes

$$\frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \qquad (30.4)$$

- For flow over a flat plate, $\frac{dp}{dx} = 0, \quad hence\ U_\infty \frac{dU_\infty}{dx} = 0$ and the governing Eq. (30.1) reduces to

$$\frac{d\delta^{**}}{dx} = \frac{\tau_w}{\rho U_\infty^2} \qquad (30.5)$$

- Again from Eq. (29.8), the momentum thickness is

$$\delta^{**} = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy \quad or\ \delta^{**} = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy$$

$$or\ \delta^{**} = \delta\int_0^1 \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right)d\eta$$

$$or\ \delta^{**} = \frac{39}{280}\delta$$

 The wall shear stress is given by

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$or\ \tau_w = \mu\left[\frac{\partial}{\delta\partial\eta}\left\{U_\infty\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\right\}\right]_{\eta=0}$$

$$or\ \tau_w = \frac{3\mu U_\infty}{2\delta}$$

- Substituting the values of δ^{**} and τ_w in Eq. (30.5) we get,

$$\frac{39}{280}\frac{d\delta}{dx} = \frac{3\mu U_\infty}{2\delta\rho U_\infty^2}$$

$$or \int \delta d\delta = \int \frac{140}{13} \frac{\mu}{\rho U_\infty} dx + C_1$$

$$or \frac{\delta^2}{2} = \frac{140}{13} \frac{\nu x}{U_\infty} + C_1 \qquad (30.6)$$

where C_1 is any arbitrary unknown constant.

- The condition at the leading edge ($at\ x = 0, \delta = 0$) yields $C_1 = 0$
  Finally we obtain,

$$\delta^2 = \frac{280}{13} \frac{\nu x}{U_\infty} \qquad (30.7)$$

$$or\ \delta = 4.54 \sqrt{\frac{\nu x}{U_\infty}}$$

$$or\ \delta = \frac{4.64 x}{\sqrt{Re_x}} \qquad (30.8)$$

- This is the value of boundary layer thickness on a flat plate. Although, the method is an approximate one, the result is found to be reasonably accurate. The value is slightly lower than the exact solution of laminar flow over a flat plate . As such, **the accuracy depends on the order of the velocity profile.** We could have have used a fourth order polynomial instead --

$$\frac{u}{U_\infty} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 \qquad (30.9)$$

- In addition to the boundary conditions in Eq. (30.3), we shall require another boundary condition at

$$y = \delta, \frac{\partial^2 u}{\partial y^2} = 0\ or\ at\ \eta = 1, \frac{\partial^2 (u/U_\infty)}{\partial \eta^2} = 0$$

This yields the constants as $a_0 = 0, a_1 = 2, a_3 = -2\ and\ a_4 = 1$.

Finally the velocity profile will be

$$\frac{u}{U_\infty} = 2\eta - 2\eta^3 + \eta^4$$

Subsequently, for a fourth order profile the growth of boundary layer is given by

$$\delta = \frac{5.83x}{\sqrt{Re_x}} \tag{30.10}$$

8.6 Turbulent Boundary Layer over Flat Plate:

Derivation of Governing Equations for Turbulent Flow

- For **incompressible flows**, the Navier-Stokes equations can be rearranged in the form

$$\rho\left[\frac{\partial u}{\partial t}+\frac{\partial (u^2)}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\nabla^2 u \tag{33.1a}$$

$$\rho\left[\frac{\partial v}{\partial t}+\frac{\partial (uv)}{\partial x}+\frac{\partial (v^2)}{\partial y}+\frac{\partial (vw)}{\partial z}\right]=-\frac{\partial p}{\partial y}+\mu\nabla^2 v \tag{33.1b}$$

$$\rho\left[\frac{\partial w}{\partial t}+\frac{\partial (uw)}{\partial x}+\frac{\partial (vw)}{\partial y}+\frac{\partial (w^2)}{\partial z}\right]=-\frac{\partial p}{\partial z}+\mu\nabla^2 w \tag{33.1c}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{33.2}$$

- Express the velocity components and pressure in terms of time-mean values and corresponding fluctuations. In **continuity equation**, this substitution and subsequent time averaging will lead to

$$\overline{\frac{\partial (\bar{u}+u')}{\partial x}}+\overline{\frac{\partial (\bar{v}+v')}{\partial y}}+\overline{\frac{\partial (\bar{w}+w')}{\partial z}}=0$$

or,

$$\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}\right)+\left(\frac{\partial \overline{u'}}{\partial x}+\frac{\partial \overline{v'}}{\partial y}+\frac{\partial \overline{w'}}{\partial z}\right)=0$$

Since, $$\frac{\partial \overline{u'}}{\partial x} = \frac{\partial \overline{v'}}{\partial y} = \frac{\partial \overline{w'}}{\partial z} = 0$$

We can write $$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \qquad (33.3a)$$

From Eqs (33.3a) and (33.2), we obtain

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad (33.3b)$$

- It is evident that **the time-averaged velocity components** and **the fluctuating velocity components**, each satisfy the continuity equation for incompressible flow.
- Imagine a two-dimensional flow in which the turbulent components are independent of the z -direction. Eventually, Eq.(33.3b) tends to

$$\frac{\partial u'}{\partial x} = -\frac{\partial v'}{\partial y} \qquad (33.4)$$

On the basis of condition (33.4), it is postulated that if at an instant there is an increase in u' in the x -direction, it will be followed by an increase in v' in the negative y -direction. In other words, $\overline{u'v'}$ **is non-zero and negative**. (see Figure 33.2)

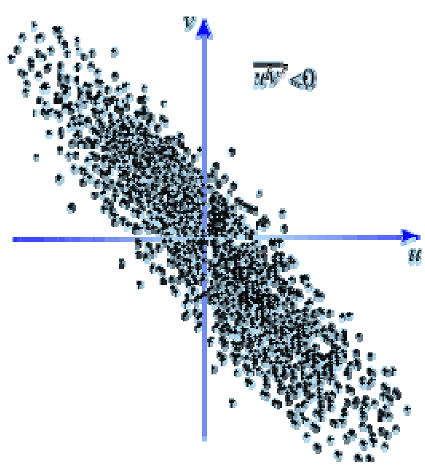


Fig 33.2 Each dot represents uv pair at an instant

- Invoking the concepts of eqn. (32.8) into the equations of motion eqn (33.1 a, b, c), we obtain expressions in terms of mean and fluctuating components. Now, forming time averages and considering the rules of averaging we discern the following. The terms which are linear, such as $\frac{\partial u'}{\partial t}$ and $\frac{\partial^2 u'}{\partial x^2}$ vanish when they are averaged [from (32.6)]. The same is true for the mixed terms like $\bar{u}\cdot u'$, or $\bar{u}\cdot v'$, but the quadratic terms in the fluctuating components remain in the equations. After averaging, they form $\overline{u'^2}$, $\overline{u'v'}$ etc.

- If we perform the aforesaid exercise on the x-momentum equation, we obtain

$$\rho\left\{\frac{\partial \bar{u}}{\partial t}+\frac{\partial u'}{\partial t}+\frac{\partial\left(\bar{u}^2+\overline{u'^2}\right)}{\partial x}+\frac{\partial\left(\overline{u\cdot v}+\overline{u'v'}\right)}{\partial y}+\frac{\partial\left(\bar{u}\cdot\bar{w}+\overline{u'w'}\right)}{\partial z}\right\}$$
$$=-\frac{\partial \bar{p}}{\partial x}-\frac{\partial p'}{\partial x}+\mu\left[\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}+\left(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}+\frac{\partial^2 u'}{\partial z^2}\right)\right]$$

using rules of time averages,

$$\frac{\partial \overline{u'}}{\partial t}=0,\ \frac{\partial \overline{p'}}{\partial x}=0,\ \frac{\partial^2 \overline{u'}}{\partial x^2}=\frac{\partial^2 \overline{u'}}{\partial y^2}=\frac{\partial^2 \overline{u'}}{\partial z^2}=0$$

We obtain

$$\rho\left\{\frac{\partial \bar{u}}{\partial t}+\frac{\partial\left(\bar{u}^2\right)}{\partial x}+\frac{\partial\left(\bar{u}\cdot\bar{v}\right)}{\partial y}+\frac{\partial\left(\bar{u}\cdot\bar{w}\right)}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}-\rho\left[\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\overline{u'v'}+\frac{\partial}{\partial z}\overline{u'w'}\right]$$

- Introducing simplifications arising out of continuity Eq. (33.3a), we shall obtain.

$$\rho\left\{\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}-\rho\left[\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\overline{u'v'}+\frac{\partial}{\partial z}\overline{u'w'}\right]$$

- Performing a similar treatment on y and z momentum equations, finally we obtain the momentum equations in the form.

In x direction,

$$\rho\left\{\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}-\rho\left[\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\overline{u'v'}+\frac{\partial}{\partial z}\overline{u'w'}\right] \quad (33.5a)$$

In y direction,

$$\rho\left\{\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial y}+\mu\nabla^2\bar{v}-\rho\left[\frac{\partial}{\partial x}\overline{u'v'}+\frac{\partial}{\partial y}\overline{v'^2}+\frac{\partial}{\partial z}\overline{v'w'}\right] \quad (33.5b)$$

In z direction,

$$\rho\left\{\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}-\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial z}+\mu\nabla^2\bar{w}-\rho\left[\frac{\partial}{\partial x}\overline{u'w'}+\frac{\partial}{\partial y}\overline{v'w'}+\frac{\partial}{\partial z}\overline{w'^2}\right] \quad (33.5c)$$

- Comments on the governing equation :
    1. The left hand side of Eqs (33.5a)-(33.5c) are essentially similar to the steady-state Navier-Stokes equations if the velocity components *u, v* and *w* are replaced by $\bar{u}$, $\bar{v}$ and $\bar{w}$.
    2. The same argument holds good for the first two terms on the right hand side of Eqs (33.5a)-(33.5c).
    3. However, the equations contain some additional terms which depend on turbulent fluctuations of the stream. **These additional terms can be interpreted as components of a stress tensor.**
- Now, the resultant surface force per unit area due to these terms may be considered as

In x direction,

$$\rho\left\{\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}+\left[\frac{\partial}{\partial x}\sigma'_{xx}+\frac{\partial}{\partial y}\tau'_{yx}+\frac{\partial}{\partial z}\tau'_{zx}\right] \quad (33.6a)$$

In y direction,

$$\rho\left\{\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial y}+\mu\nabla^2\bar{v}+\left[\frac{\partial}{\partial x}\tau'_{xy}+\frac{\partial}{\partial y}\sigma'_{yy}+\frac{\partial}{\partial z}\tau'_{zy}\right] \quad (33.6b)$$

In z direction,

$$\rho\left\{\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right\}=-\frac{\partial \bar{p}}{\partial z}+\mu\nabla^2\bar{w}+\left[\frac{\partial}{\partial x}\tau'_{xz}+\frac{\partial}{\partial y}\tau'_{yz}+\frac{\partial}{\partial z}\sigma'_{zz}\right] \quad (33.6c)$$

- Comparing Eqs (33.5) and (33.6), we can write

$$\begin{bmatrix}\sigma'_{xx} & \tau'_{xy} & \tau'_{xz}\\ \tau'_{xy} & \sigma'_{yy} & \tau'_{yz}\\ \tau'_{xz} & \tau'_{yz} & \sigma'_{zz}\end{bmatrix}=-\rho\begin{bmatrix}\overline{u'^2} & \overline{u'v'} & \overline{u'w'}\\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'}\\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2}\end{bmatrix} \quad (33.7)$$

- It can be said that the mean velocity components of turbulent flow satisfy the same Navier-Stokes equations of laminar flow. However, for the turbulent flow, the laminar stresses must be increased by additional stresses which are given by the stress tensor (33.7). These additional stresses are known as **apparent stresses of turbulent flow** or **Reynolds stresses** . Since turbulence is considered as eddying motion and the aforesaid additional stresses are added to the viscous stresses due to mean motion in order to explain the complete stress field, it is often said that the apparent stresses are caused by eddy viscosity . The total stresses are now

$$\left.\begin{aligned}\sigma_{xx}&=-\bar{p}-2\mu\frac{\partial \bar{u}}{\partial x}-\overline{\rho u'^2}\\ \tau_{xy}&=\mu\left(\frac{\partial \bar{u}}{\partial y}+\frac{\partial \bar{v}}{\partial x}\right)-\overline{\rho u'v'}\end{aligned}\right] \quad (33.8)$$

and so on. The apparent stresses are much larger than the viscous components, and the viscous stresses can even be dropped in many actual calculations.

Turbulent Boundary Layer Equations

- For a two-dimensional flow (w = 0)over a flat plate, the thickness of turbulent boundary layer is assumed to be much smaller than the axial length and the **order of magnitude analysis** may be applied. As a consequence, the following inferences are drawn:

$$(a) \quad \frac{\partial \bar{p}}{\partial y}=0,$$

(b) $\frac{\partial \bar{p}}{\partial x} = \frac{d\bar{p}}{dx}$

(c) $\frac{\partial^2 \bar{u}}{\partial x^2} << \frac{\partial^2 \bar{u}}{\partial y^2}$,

(d) $\frac{\partial}{\partial x}\left(-\rho\overline{u'^2}\right) << \frac{\partial}{\partial y}\left(-\rho\overline{u'v'}\right)$

- The turbulent boundary layer equation together with the equation of continuity becomes

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \qquad (33.9)$$

-

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\cdot\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y}\left\{\nu\frac{\partial \bar{u}}{\partial y} - \overline{u'v'}\right\} \qquad (33.10)$$

- A comparison of Eq. (33.10) with laminar boundary layer Eq. (23.10) depicts that: *u, v* and *p* are replaced by the time average values $\bar{u}, \bar{v}$ and $\bar{p}$, and laminar viscous force per unit volume $\frac{\partial(\tau_l)}{\partial y}$ is replaced by $\frac{\partial}{\partial y}(\tau_l + \tau_t)$ where $\tau_l = \mu\frac{\partial \bar{u}}{\partial y}$ is the laminar shear stress and $\tau_t = -\rho\overline{u'v'}$ is the turbulent shear stress.

Boundary Conditions

- All the components of apparent stresses vanish at the solid walls and only stresses which act near the wall are the viscous stresses of laminar flow. The boundary conditions, to be satisfied by the mean velocity components, are similar to laminar flow.
- A very thin layer next to the wall behaves like a near wall region of the laminar flow. This layer is known as **laminar sublayer** and its velocities are such that the viscous

forces dominate over the inertia forces. No turbulence exists in it (see Fig. 33.3).

- For a developed turbulent flow over a flat plate, in the near wall region, inertial effects are insignificant, and we can write from Eq.33.10,

$$\nu\frac{\partial^2\bar{u}}{\partial y^2}-\frac{\partial\left(\overline{u'v'}\right)}{\partial y}=0$$

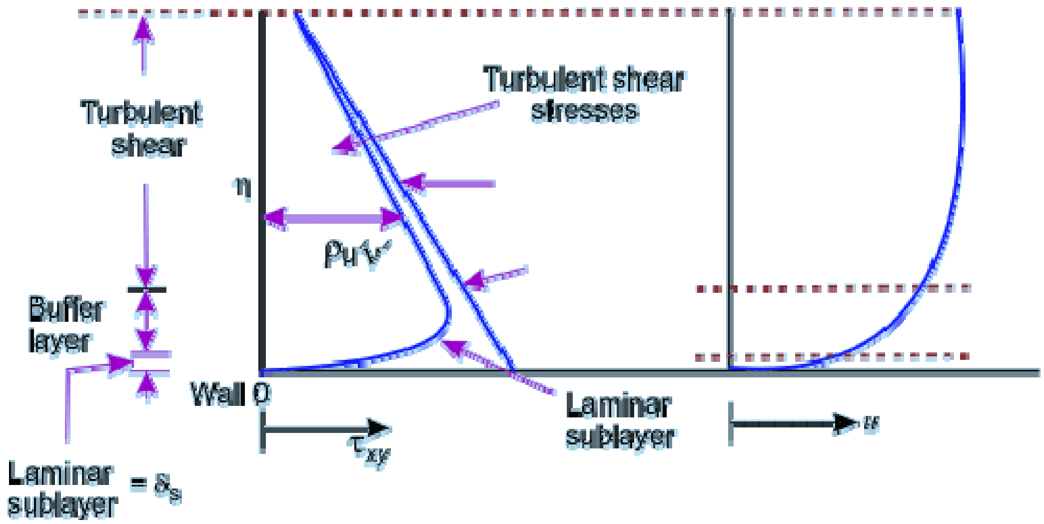


Fig 33.3 Different zones of a turbulent flow past a wall

which can be integrated as , $\frac{\nu\partial\bar{u}}{\partial y}-\overline{u'v'}$ =constant

- We know that the fluctuating components, do not exist near the wall, the shear stress on the wall is purely viscous and it follows

$$\nu\frac{\partial\bar{u}}{\partial y}\bigg|_{y=0}=\frac{\tau_w}{\rho}$$

However, the wall shear stress in the vicinity ofthe laminar sublayer is estimated as

$$\tau_w=\mu\left[\frac{U_s-0}{\delta_s-0}\right]=\mu\frac{U_s}{\delta_s} \qquad (33.11a)$$

where U_s is the fluid velocity at the edge of the sublayer. The flow in the sublayer is specified by a velocity scale (characteristic of this region).

- We define the **friction velocity**,

$$u_\tau = \left[\frac{\tau_w}{\rho}\right]^{1/2} \qquad (33.11b)$$

as our velocity scale. Once u_τ is specified, the structure of the sub layer is specified. It has been confirmed experimentally that the turbulent intensity distributions are scaled with u_τ. For example, maximum value of the $\overline{u'^2}$ is always about $8u_\tau^2$. The relationship between u_τ and the U_s can be determined from Eqs (33.11a) and (33.11b) as

$$u_\tau^2 = \nu\frac{U_s}{\delta_s}$$

Let us assume $U_s = \bar{C}U_\infty$. Now we can write

$$u_\tau^2 = \bar{C}\nu\frac{U_\infty}{\delta_s} \qquad (33.12a)$$

where $\bar{C}$ is a proportionality constant

or

$$\frac{\delta_s u_\tau}{\nu} = \bar{C}\left[\frac{U_\infty}{u_\tau}\right] \qquad (33.12b)$$

Hence, a non-dimensional coordinate may be defined as, $\eta = \frac{y u_\tau}{\nu}$ which will help us estimating different zones in a turbulent flow. **The thickness** of **laminar sublayer or viscous sublayer is considered to be** $\eta \approx 5$.

Turbulent effect starts in the zone of $\eta > 5$ and in a zone of $5 < \eta < 70$, laminar and turbulent motions coexist. This domain is termed as **buffer zone**. Turbulent effects far outweight the laminar effect in the zone beyond $\eta = 70$ and this regime is termed as turbulent core .

- For flow over a flat plate, the turbulent shear stress ($-\rho \overline{u'v'}$) is constant throughout in the y direction and this becomes equal to τ_w at the wall. In the event of flow through a channel, the turbulent shear stress ($-\rho \overline{u'v'}$) varies with *y* and it is possible to write

$$\frac{\tau_t}{\tau_w} = \frac{\zeta}{h} \qquad (33.12c)$$

where the channel is assumed to have a height ***2h*** and ζ is the distance measured from the centreline of the channel $(= h - y)$. Figure 33.1 explains such variation of turbulent stress.

Shear Stress Models

- In analogy with the coefficient of viscosity for laminar flow, J. Boussinesq introduced a **mixing coefficient** μ_T for the Reynolds stress term, defined as

$$\tau_t = -\rho\overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$

- Using μ_T the shearing stresses can be written as

$$\tau_l = \rho\nu \frac{\partial \bar{u}}{\partial y}, \tau_t = \mu_t \frac{\partial \bar{u}}{\partial y} = \rho\nu_t \frac{\partial \bar{u}}{\partial y}$$

such that the equation

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}.\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left\{\nu\frac{\partial \bar{u}}{\partial y} - \overline{u'v'}\right\}$$

may be written as

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{1}{\rho}.\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left\{(\nu + \nu_t)\frac{\partial \bar{u}}{\partial y}\right\} \qquad (33.13)$$

The term ν_t is known as **eddy viscosity** and the model is known as **eddy viscosity model** .

- Unfortunately the value of ν_t is not known. The term ν is a property of the fluid whereas ν_t is attributed to random fluctuations and is not a property of the fluid. However, it is necessary to find out empirical relations between ν_t, and the mean velocity. The following section discusses relation between the aforesaid apparent or eddy viscosity and the mean velocity components

Prandtl's Mixing Length Hypothesis

- Consider a fully developed turbulent boundary layer . The stream wise mean velocity varies only from streamline to streamline. The main flow direction is assumed parallel to the x-axis (Fig. 33.4).

- The time average components of velocity are given by $\bar{u} = \bar{u}(y), \bar{v} = 0, \bar{w} = 0$. The fluctuating component of transverse velocity v'transports mass and momentum across a plane at y_1 from the wall. The shear stress due to the fluctuation is given by

$$\tau_t = -\rho\overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y} \qquad (33.14)$$

- Fluid, which comes to the layer y_1 from a layer $(y_1 - l)$ has a positive value of v'. If the lump of fluid retains its original momentum then its velocity at its current location y_1 is smaller than the velocity prevailing there. The difference in velocities is then

$$\Delta u_1 = \bar{u}(y_1) - \bar{u}(y_1 - l) \approx l\left(\frac{\partial \bar{u}}{\partial y}\right)_{y_1} \qquad (33.15)$$

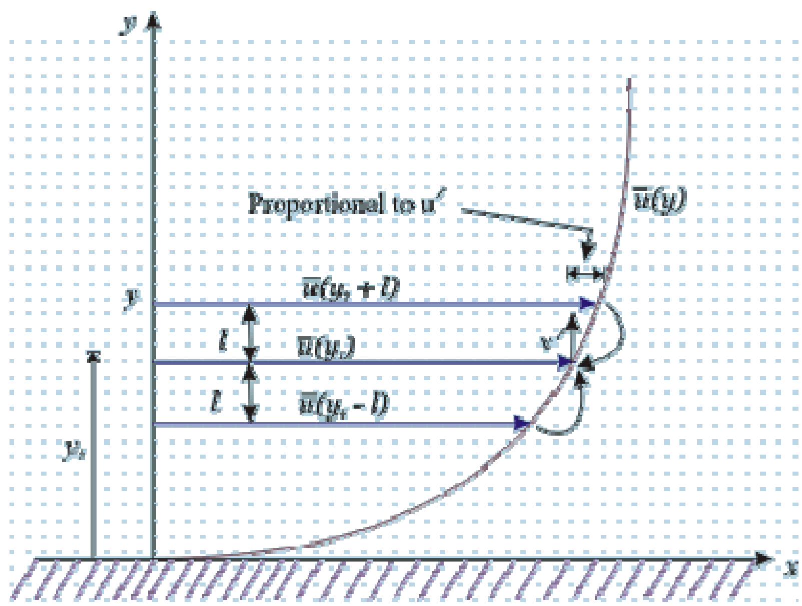


Fig. 33.4 One-dimensional parallel flow and Prandtl's mixing length hypothesis

The above expression is obtained by expanding the function $\bar{u}(y_1 - l)$in a Taylor series and neglecting all higher order terms and higher order derivatives. l is a small length scale known as Prandtl's mixing length . Prandtl proposed that the transverse displacement of any fluid particle is, on an average, 'l' .

- Consider another lump of fluid with a negative value of v'. This is arriving at $y_1$from $(y_1 + l)$. If this lump retains its original momentum, its mean velocity at the current lamina y_1 will be somewhat more than the original mean velocity of y_1. This difference is given by

$$\Delta u_2 = \bar{u}(y_1 + l) - \bar{u}(y_1) \approx l\left(\frac{\partial \bar{u}}{\partial y}\right)_{y_1}$$

(33.16)

- The velocity differences caused by the transverse motion can be regarded as the turbulent velocity components at y_1.
- We calculate the time average of the absolute value of this fluctuation as

$$\overline{|u'|} = \frac{1}{2}\left(|\Delta u_1| + |\Delta u_2|\right) = l\left|\left(\frac{\partial \bar{u}}{\partial y}\right)_{y_1}\right|$$

(33.17)

- Suppose these two lumps of fluid meet at a layer y_1The lumps will collide with a velocity $2u'$ and diverge. This proposes the possible existence of transverse velocity component in both directions with respect to the layer at y_1. Now, suppose that the two lumps move away in a reverse order from the layer $y_1$with a velocity $2u'$. The empty space will be filled from the surrounding fluid creating transverse velocity components which will again collide at y_1. Keeping in mind this argument and the physical explanation accompanying Eqs (33.4), we may state that

$$\overline{|v'|} \sim \overline{|u'|}$$

or, $$\overline{|v'|} = (const)\overline{|u'|} = (const)\, l\left|\left(\frac{\partial \bar{u}}{\partial y}\right)\right|$$

along with the condition that the moment at which u' is positive, v' is more likely to be negative and conversely when u' is negative. Possibly, we can write at this stage

$$\overline{u'v'} = -C_1 |u'||v'|$$

$$\overline{u'v'} = -C_2 l^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

(33.18)

where C_1 and C_2 are different proportionality constants. However, the constant C_2 can now be included in still unknown mixing length and Eg. (33.18) may be rewritten as

$$\overline{u'v'} = -l^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

- For the expression of turbulent shearing stress τ_t we may write

$$\mu_t = \rho l^2 \left|\frac{\partial u}{\partial y}\right|$$

(33.19)

- After comparing this expression with the eddy viscosity Eg. (33.14), we may arrive at a more precise definition,

$$\tau_t = \rho l^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \left(\frac{\partial \bar{u}}{\partial y}\right) = \mu_t \frac{\partial \bar{u}}{\partial y}$$

(33.20a)

where the apparent viscosity may be expressed as

$$\mu_t = \rho l^2 \left|\frac{\partial \bar{u}}{\partial y}\right|$$

(33.20b)

and the apparent kinematic viscosity is given by

$$\nu_t = l^2 \left|\frac{\partial \bar{u}}{\partial y}\right|$$

(33.20c)

- The decision of expressing one of the velocity gradients of Eq. (33.19) in terms of its modulus as $\left|\frac{\partial \bar{u}}{\partial y}\right|$ was made in order to assign a sign to τ_t according to the sign of $\frac{\partial \bar{u}}{\partial y}$.
- Note that the apparent viscosity and consequently,the mixing length are not properties of fluid. They are dependent on turbulent fluctuation.
- But how to determine the value of "l" the mixing length? Several correlations, using experimental results for τ_t have been proposed to determine l.

  However, so far the most widely used value of mixing length in the regime of isotropic turbulence is given by

$$l = \chi y$$

(33.21)

where y is the distance from the wall and χ is known as **von Karman constant** (≈ 0.4).

8.7 Skin Friction Drag:

Wall Shear Stress

- With the profile known, wall shear can be evaluated as

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0}$$

Now, $\dfrac{\partial u}{\partial y} = U_\infty f''(\eta)\dfrac{\partial \eta}{\partial y}$

or $$\tau_w = \mu U_\infty f''(\eta)\frac{\partial \eta}{\partial y}\bigg|_{y=0}$$

$$= \mu U_\infty H \frac{\partial \eta}{\partial y}\bigg|_{y=0}$$

or $$\tau_w = \mu U_\infty \times 0.3326 \times \frac{1}{\sqrt{(\nu x)/U_\infty}}$$

$[f''(0) = 0.3326]$ from Table 28.1

$$\tau_w = \frac{0.332\rho U_\infty^2}{\sqrt{\mathrm{Re}_x}} \qquad (29.1a)$$

(Wall Shear Stress)

and the local skin friction coefficient is $C_f = \dfrac{\tau_w}{1/2\rho U_\infty^2}$

- Substituting from (29.1a) we get

$$C_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \qquad (29.1b)$$

(Skin Friction Coefficient)

-
- In 1951, Liepmann and Dhawan , measured the shearing stress on a flat plate directly. Their results showed a striking confirmation of Eq. (29.1).
- Total frictional force per unit width for the plate of length L is

$$F = \int_0^L \tau_w \, dx$$

$$F = \int_0^L \frac{0.332 \rho U_\infty^2}{\sqrt{U_\infty / \nu}} \frac{dx}{\sqrt{x}}$$

or

$$F = \left[\frac{0.332 \rho U_\infty^2}{\sqrt{U_\infty / \nu}} \times \frac{x^{1/2}}{1/2} \right]_0^L$$

or

$$F = 0.664 \times \rho U_\infty^2 \sqrt{\frac{\nu L}{U_\infty}} \qquad (29.2)$$

and the average skin friction coefficient is

$$\overline{C_f} = \frac{F}{1/2\left(\rho U_\infty^2 L\right)} = \frac{1.328}{\sqrt{\mathrm{Re}_L}} \qquad (29.3)$$

where, $\mathrm{Re}_L = U_\infty L / \nu$.

For a flat plate of length L in the streamwise direction and width w perpendicular to the flow, the Drag D would be

$$D = F(2wL) = 0.664(2wL)\rho U_\infty^2 \left(\frac{\nu L}{U_\infty}\right)^{1/2} = 1.328wL\left(\frac{\rho \mu U_\infty^3}{L}\right)^{1/2} \qquad (29.4)$$

8.8 Boundary Layer Control:

Seperation of Boundary Layer

- It has been observed that the **flow is reversed at the vicinity of the wall** under certain conditions.
- The phenomenon is termed as **separation of boundary layer**.
- Separation takes place **due to excessive momentum loss near the wall in a boundary layer trying to move downstream against increasing pressure, i.e.,** $\frac{dp}{dx} > 0$ **, which is called *adverse pressure gradient.***
- Figure 29.2 shows the flow past a circular cylinder, in an infinite medium.
  1. Up to $\theta = 90^\circ$, the flow area is like a constricted passage and the flow behaviour is like that of a nozzle.
  2. Beyond $\theta = 90^\circ$ the flow area is diverged, therefore, the flow behaviour is much similar to a diffuser

This dictates the inviscid pressure distribution on the cylinder which is shown by a firm line in Fig. 29.2.

Here

p_∞ **: pressure in the free stream**

U_∞ **: velocity in the free stream and**

p ***:* is the local pressure on the cylinder.**

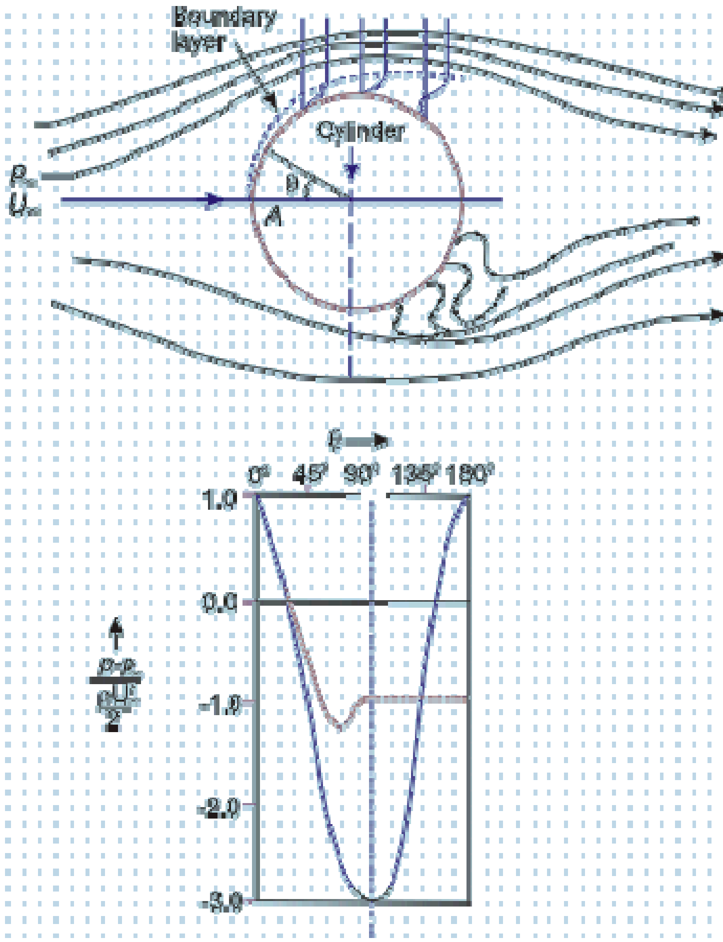


Fig. 29.2 Flow separation and formation of wake behind a circular cylinder

- Consider the forces in the flow field.
  In the **inviscid region,**
  1. **Until** $\theta = 90°$ the pressure force and the force due to streamwise acceleration i.e. inertia forces are acting in the same direction (**pressure gradient** being **negative/favourable**)
  2. **Beyond** $\theta = 90°$, the **pressure gradient is positive or adverse**. Due to the adverse pressure gradient the pressure force and the force due to acceleration will be opposing each other in the in viscid zone of this part.
-

  So long as no viscous effect is considered, the situation does not cause any sensation.
  In the **viscid region** (near the solid boundary),

  1. **Up to** $\theta = 90°$**, th**e viscous force opposes the combined pressure force and the force due to acceleration. Fluid particles overcome this viscous resistance **due to continuous conversion of pressure force into kinetic energy.**

2. Beyond $\theta = 90^\circ$, within the viscous zone, the flow structure becomes different. It is seen that the force due to acceleration is opposed by both the viscous force and pressure force.

- Depending upon the magnitude of adverse pressure gradient, **somewhere around $\theta = 90^\circ$, the fluid particles, in the boundary layer are separated from the wall** and driven in the upstream direction. However, the far field external stream pushes back these separated layers together with it and develops **a broad pulsating wake behind the cylinder**.
- **The mathematical explanation of flow-separation :** The point of separation may be defined as the limit between forward and reverse flow in the layer very close to the wall, i.e., at the point of separation

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \qquad (29.16)$$

This means that the shear stress at the wall, $\tau_w = 0$. But at this point, the adverse pressure continues to exist and at the downstream of this point the flow acts in a reverse direction resulting in a back flow.

- We can also explain flow separation using the argument about the second derivative of velocity *u* at the wall. From the dimensional form of the momentum at the wall, where $u = v = 0$, we can write

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \frac{1}{\mu}\frac{dp}{dx} \qquad (29.17)$$

- Consider the situation due to a **favourable pressure gradient** where $\frac{dp}{dx} < 0$ we have,

1. $\left(\partial^2 u / \partial y^2\right)_{wall} < 0$. (From Eq. (29.17))
2. **As we proceed towards the free stream, the velocity *u* approaches U_∞ asymptotically,** so $\partial u / \partial y$ decreases at a continuously lesser rate in *y* direction.
3. This means that $\partial^2 u / \partial y^2$ remains less than zero near the edge of the boundary layer.
4. The curvature of a velocity profile $\partial^2 u / \partial y^2$ is always negative as shown in (Fig. 29.3a)

- Consider the case of **adverse pressure gradient**, $\partial p/\partial x > 0$
  1. At the boundary, the curvature of the profile must be positive (since $\partial p/\partial x > 0$).
  2. Near the interface of boundary layer and free stream the previous argument regarding $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ still holds good and the curvature is negative.
  3. Thus we observe that for an adverse pressure gradient, there must exist a point for which $\partial^2 u/\partial y^2 = 0$. This point is known as *point of inflection* of the velocity profile in the boundary layer as shown in Fig. 29.3b
  4. However, point of separation means $\partial u/\partial y = 0$ at the wall.
  5. $\partial^2 u/\partial y^2 > 0$ at the wall since separation can only occur due to adverse pressure gradient. But we have already seen that at the edge of the boundary layer, $\partial^2 u/\partial y^2 < 0$. It is therefore, clear that **if there is a point of separation, there must exist a point of inflection in the velocity profile.**

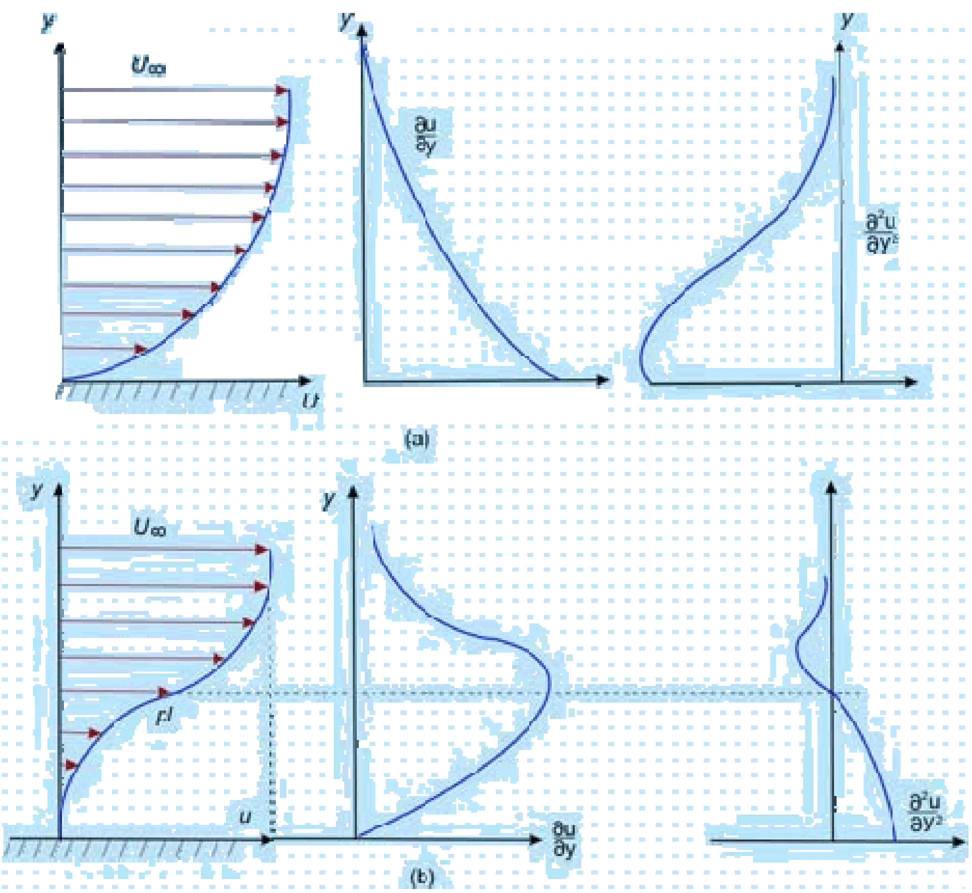


Fig. 29.3 Velocity distribution within a boundary layer

(a) Favourable pressure gradient, $\frac{dp}{dx} < 0$

(b) adverse pressure gradient, $\frac{dp}{dx} > 0$

1. Let us reconsider the flow past a circular cylinder and continue our **discussion on the wake behind a cylinder.** The pressure distribution which was shown by the firm line in Fig. 21.5 is obtained from the potential flow theory. However. somewhere near $\theta = 90^\circ$ **(in experiments it has been observed to be at $\theta = 81^\circ$) . the boundary layer detaches itself from the wall.**
2. Meanwhile, **pressure in the wake remains close to separation-point-pressure** since the eddies (formed as a consequence of the retarded layers being carried together with the upper layer through the action of shear) cannot convert rotational kinetic energy into pressure head. The actual pressure distribution is shown by the dotted line in Fig. 29.3.
3. Since the **wake zone pressure is less than that of the forward stagnation point** (pressure at point A in Fig. 29.3), the cylinder experiences a drag force which is basically attributed to the pressure difference.
   The drag force, brought about by the pressure difference is known as *form drag* whereas the shear stress at the wall gives rise to *skin friction drag*. Generally, these two drag forces together are responsible for resultant drag on a body

Control Of Boundary Layer Separation -

- The total drag on a body is attributed to form drag and skin friction drag. In some flow configurations, the contribution of form drag becomes significant.
- In order **to reduce the form drag, the boundary layer separation** should be **prevented or delayed** so that **better pressure recovery takes place** and the form drag is reduced considerably. There are some popular methods for this purpose which are stated as follows.
  i. By giving the profile of the body a streamlined shape(as shown in *Fig. 31.2*).
     1. This has an elongated shape in the rear part to reduce the magnitude of the pressure gradient.
     2. The optimum contour for a streamlined body is the one for which the wake zone is very narrow and the form drag is minimum.

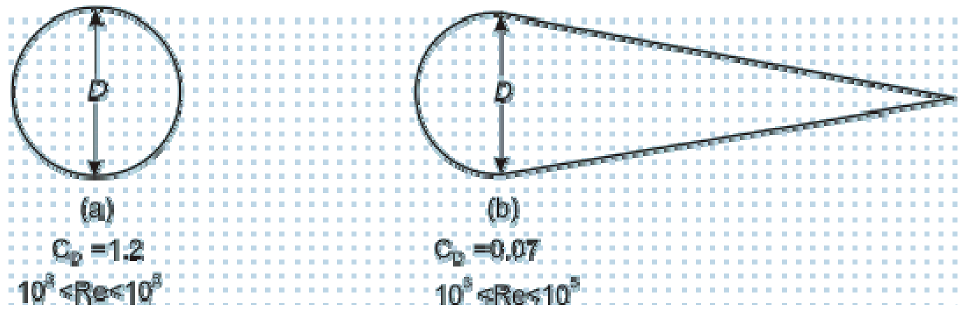


Fig. 31.2 Reduction of drag coefficient (C_D) by giving the profile a streamlined shape

ii. **The injection of fluid through porous wall can also control the boundary layer separation. This is generally accomplished by** blowing high energy fluid particles tangentially from the location where separation would have taken place otherwise. This is shown in Fig. 31.3.

1. The **injection of fluid promotes turbulence**
2. This **increases skin friction.** But the **form drag is reduced** considerably due to suppression of flow separation
3. The reduction in form drag is quite significant and **increase in skin friction drag can be ignored.**

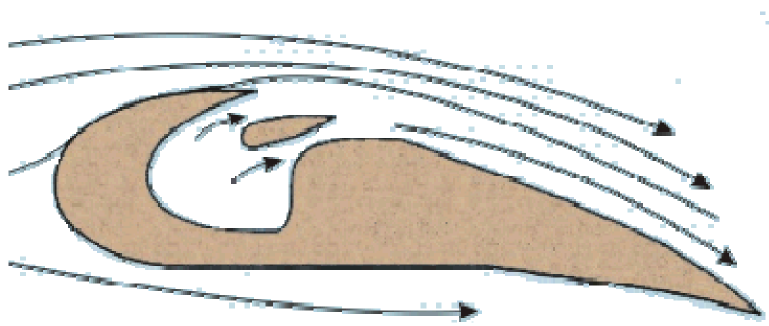


Fig. 31.3 Boundary layer control by blowing

Chapter 9
Dimensional Analysis and Similitude

9.1 Buckingham's Theorem:

The Application of Dynamic Similarity - The Dimensional Analysis

The concept:

A physical problem may be characterised by a group of dimensionless similarity parameters or variables rather than by the original dimensional variables.

This gives a clue to the reduction in the number of parameters requiring separate consideration in an experimental investigation.

For an **example,** if the Reynolds number $Re = \rho V D_h / \mu$ is considered as the independent variable, in case of a flow of fluid through a closed duct of hydraulic diameter D_h, then a change in Re may be caused through a change in flow velocity V only. Thus a range of Re can be covered simply by the variation in V without varying other independent dimensional variables ρ,D_h and μ.

In fact, the variation in the Reynolds number physically implies the variation in any of the dimensional parameters defining it, though the change in Re, may be obtained through the variation in anyone parameter, say the velocity V.

A number of such **dimensionless parameters** in relation to dynamic similarity are shown in Table 5.1. Sometimes it becomes diffcult to derive these parameters straight forward from an estimation of the representative order of magnitudes of the forces involved. An alternative **method of determining these dimensionless parameters by a mathematical technique is known as dimensional analysis** .

The Technique:

The requirement of dimensional homogeneity imposes conditions on the quantities involved in a physical problem, and these restrictions, placed in the form of an algebraic function by the requirement of dimensional homogeneity, play the central role in dimensional analysis.

There are two existing approaches;

- one due to Buckingham known as **Buckingham's pi theorem**
- other due to Rayleigh known as **Rayleigh's Indicial method**

In our next slides we'll see few examples of the dimensions of physical quantities.

Dimensions of Physical Quantities

All physical quantities are expressed by magnitudes and units.

For ***example*** , the velocity and acceleration of a fluid particle are 8m/s and $10 m/s^2$ respectively. Here the dimensions of velocity and acceleration are ms^{-1} and ms^{-2} respectively.

In SI (System International) units, the primary physical quantities which are assigned base dimensions are the mass, length, time, temperature, current and luminous intensity. Of these, the first four are used in fluid mechanics and they are symbolized as M (mass), L (length), T (time), and θ (temperature).

- Any physical quantity can be expressed in terms of these primary quantities by using the basic mathematical definition of the quantity.
- The resulting expression is known as the dimension of the quantity.

Let us take some **examples:**

1. **Dimension of Stress**

Shear stress τ is defined as force/area. Again, force = mass × acceleration

Dimensions of acceleration = Dimensions of velocity/Dimension of time.

$$= \frac{\text{Dimension of Distance}}{(\text{Dimension of Time})^2}$$

$$= \frac{L}{T^2}$$

Dimension of area = (Length)^2 =L^2

Hence, dimension of shear stress

$$\tau = \left(ML/T^2\right)/L^2 = ML^{-1}T^{-2} \qquad (19.1)$$

2. **Dimension of Viscosity**

Consider Newton's law for the definition of viscosity as

$$\tau = \mu du/dy$$

or,

$$\mu = \frac{\tau}{(du/dy)}$$

The dimension of velocity gradient du/dy can be written as **dimension of du/dy= dimension of u/dimension of y = (L / T)/L = T** $^{-1}$

The dimension of shear stress τ is given in Eq. (19.1).

Hence dimension of

$$\mu = \frac{\text{Dimension of } \tau}{\text{Dimension of } du/dy} = \frac{ML^{-1}T^{-2}}{T^{-1}}$$

$$= ML^{-1}T^{-1}$$

Dimensions of Various Physical Quantities in Tabular Format

Physical Quantity	Dimension
Mass	M
Length	L
Time	T
Temperature	θ
Velocity	LT^{-1}
Angular velocity	T^{-1}
Acceleration	LT^{-2}
Angular Acceleration	T^{-2}
Force, Thrust, Weight	MLT^{-2}

Stress, Pressure	$ML^{-1}T^{-2}$
Momentum	MLT^{-1}
Angular Momentum	ML^2T^{-1}
Moment, Torque	ML^2T^{-2}
Work, Energy	ML^2T^{-2}
Power	ML^2T^{-3}
Stream Function	L^2T^{-1}
Vorticity, Shear Rate	T^{-1}
Velocity Potential	L^2T^{-1}
Density	ML^{-3}
Coefficient of Dynamic Viscosity	$ML^{-1}T^{-1}$
Coefficient of Kinematic Viscosity	L^2T^{-1}
Surface Tension	MT^{-2}
Bulk Modulus of Elasticity	$ML^{-1}T^{-2}$

Buckingham's Pi Theorem

Assume, a physical phenomenon is described by **m number of independent variables like x_1 , x_2 , x_3 , ..., x_m**

The phenomenon may be expressed analytically by an implicit functional relationship of the controlling variables as

$$f(x_1, x_2, x_3, \ldots\ldots, x_m) = 0 \quad (19.2)$$

Now if **n be the number of fundamental dimensions like mass, length, time, temperature etc** ., involved in these m variables, then according to Buckingham's p theorem -

The phenomenon can be described in terms of (m - n) independent dimensionless groups like π_1 ,π_2 , ..., π_{m-n} , where p terms, represent the dimensionless parameters and consist of different combinations of a number of dimensional variables out of the m independent variables defining the problem.

Therefore. the analytical version of the phenomenon given by Eq. (19.2) can be reduced to

$$\boxed{F(\pi_1, \pi_2, \ldots\ldots\ldots, \pi_{m-n}) = 0} \quad (19.3)$$

according to Buckingham's pi theorem

- This physically implies that the **phenomenon** which is basically described by m independent dimensional variables, **is ultimately controlled by (m-n) independent dimensionless parameters known as π terms.**

Alternative Mathematical Description of (π) Pi Theorem

A physical problem described by m number of variables involving n number of fundamental dimensions (n < m) leads to a system of n linear algebraic equations with m variables of the form

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots a_{1m}x_m = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots a_{2m}x_m = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots a_{nm}x_m = b_n \quad (19.4)$$

or in a matrix form,

$\boxed{Ax = b}$ (click the equation to follow hyperlink) (19.5) where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \\ x_m \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \\ b_m \end{bmatrix}$$

9.2 Non Dimensional Groups:

Determination of π terms

- A group of n (n = number of fundamental dimensions) variables out of m (m = total number of independent variables defining the problem) variables is first chosen to form a basis so that all n dimensions are represented . These n variables are referred to as repeating variables.

- Then the p terms are formed by the product of these repeating variables raised to arbitrary unknown integer exponents and anyone of the excluded (m -n) variables.

*For **example*** , if x_1 x_2 ...x_n are taken as the repeating variables. Then

$$\pi_2 = x_1^{a_1} x_2^{a_2} \ldots\ldots x_n^{a_n} x_{n+2}$$

$$\pi_1 = x_1^{a_1} x_2^{a_2} \ldots\ldots x_n^{a_n} x_{n+1}$$

....

$$\pi_{m-n} = x_1^{a_1} x_2^{a_2} \ldots\ldots x_n^{a_n} x_m$$

- The sets of integer exponents **a_1, a_2 . . . a_n are different for each p term.**
- Since p terms are dimensionless, it requires that when all the variables in any p term are expressed in terms of their fundamental dimensions, the exponent of all the fundamental dimensions must be zero.

- This leads to a system of n linear equations in a, a_2 . . . a_n which gives a unique solution for the exponents. This gives the values of a_1 a_2 . . . a_n for each p term and hence the p terms are uniquely defined.

In selecting the repeating variables, the following points have to be considered:

1. The repeating variables must include among them all the n fundamental dimensions, not necessarily in each one but collectively.
2. The dependent variable or the output parameter of the physical phenomenon should not be included in the repeating variables.

No physical phenomena is represented when -

- **m < n** because there is no solution **and**

- **m = n** because there is a unique solution of the variables involved and hence all the parameters have fixed values.

. **Therefore all feasible phenomena are defined with m > n .**

- **When m = n + 1,** then, according to the Pi theorem, the number of pi term is one and the phenomenon can be expressed as

$$f(\pi_1) = 0$$

where, the non-dimensional term π_1 is some specific combination of n + 1 variables involved in the problem.

When m > n+ 1 ,

1. the number of π terms are more than one.
2. A number of choices regarding the repeating variables arise in this case.

Again, it is true that if one of the repeating variables is changed, it results in a different set of π terms. Therefore the interesting question is **which set of repeating variables is to be chosen** , to arrive at the correct set of π terms to describe the problem. The **answer to this question lies in the fact that different sets of π terms resulting from the use of different sets of repeating variables are not independent. Thus, anyone of such interdependent sets is meaningful in describing the same physical phenomenon.**

From any set of such π terms, one can obtain the other meaningful sets from some combination of the π terms of the existing set without altering their total numbers (m-n) as fixed by the Pi theorem.

9.3 Geometric, Kinematic and Dynamic Similarity:

Principles of Physical Similarity - An Introduction

Laboratory tests are usually carried out under altered conditions of the operating variables from the actual ones in practice. These variables, in case of experiments relating to fluid flow, are pressure, velocity, geometry of the working systems and the physical properties of the working fluid.

The pertinent questions arising out of this situation are:

1. How to apply the test results from laboratory experiments to the actual problems?
2. Is it possible, to reduce the large number of experiments to a lesser one in achieving the same objective?

Answer of the above two questions lies in the principle of physical similarity. This principle is useful for the following cases:

1. To apply the results taken from tests under one set of conditions to another set of conditions

and

2. To predict the influences of a large number of independent operating variables on the performance of a system from an experiment with a limited number of operating variables.

Concept and Types of Physical Similarity

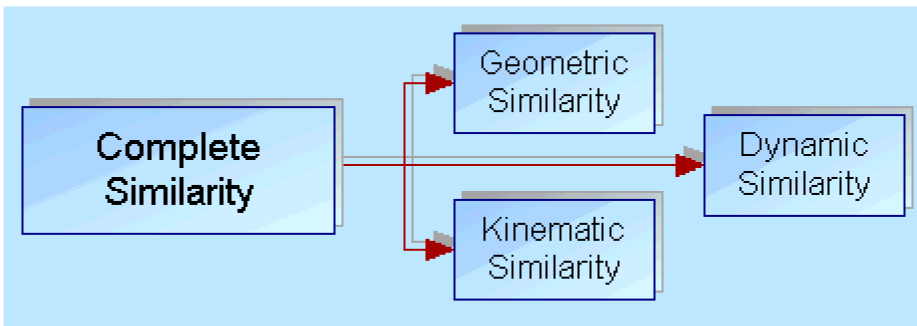
The primary and fundamental requirement for the **physical similarity** between two problems is that the **physics of the problems must be the same**.

For an example, two flows: one governed by viscous and pressure forces while the other by gravity force cannot be made physically similar. Therefore, the laws of similarity have to be sought between problems described by the same physics.

Definition of physical similarity as a general proposition.

Two systems, described by the same physics, operating under different sets of conditions are said to be physically similar in respect of certain specified physical quantities; when the ratio of corresponding magnitudes of these quantities between the two systems is the same everywhere.

In the field of mechanics, there are three types of similarities which constitute the complete similarity between problems of same kind.



Geometric Similarity : If the specified physical quantities are geometrical dimensions, the similarity is called Geometric Similarty,

Kinematic Similarity : If the quantities are related to motions, the similarity is called Kinematic Similarity

Dynamic Similarity : If the quantities refer to forces, then the similarity is termed as Dynamic Similarity.

Geometric Similarity

- Geometric Similarity implies the similarity of shape such that, the **ratio of any length in one system to the corresponding length in other system is the same everywhere**.
- This ratio is usually known as **scale factor**.

Therefore, geometrically similar objects are similar in their shapes, i.e., proportionate in their physical dimensions, but differ in size.

In investigations of physical similarity,

- the full size or **actual scale systems** are known as **prototypes**
- the **laboratory scale systems** are referred to as **models**
- use of the same fluid with both the prototype and the model is not necessary

- model need not be necessarily smaller than the prototype. The flow of fluid through an injection nozzle or a carburettor , for example, would be more easily studied by using a model much larger than the prototype.

- the model and prototype may be of identical size, although the two may then differ in regard to other factors such as velocity, and properties of the fluid.

If l_1 and l_2 are the two characteristic physical dimensions of any object, then the requirement of geometrical similarity is

$$\frac{l_{1m}}{l_{1p}} = \frac{l_{2m}}{l_{2p}} = l_r$$

(**model ratio**)

(The second suffices *m* and *p* refer to model and prototype respectively) where l_r is the scale factor or sometimes known as the model ratio. Figure 5.1 shows three pairs of geometrically similar objects, namely, a right circular cylinder, a parallelopiped, and a triangular prism.

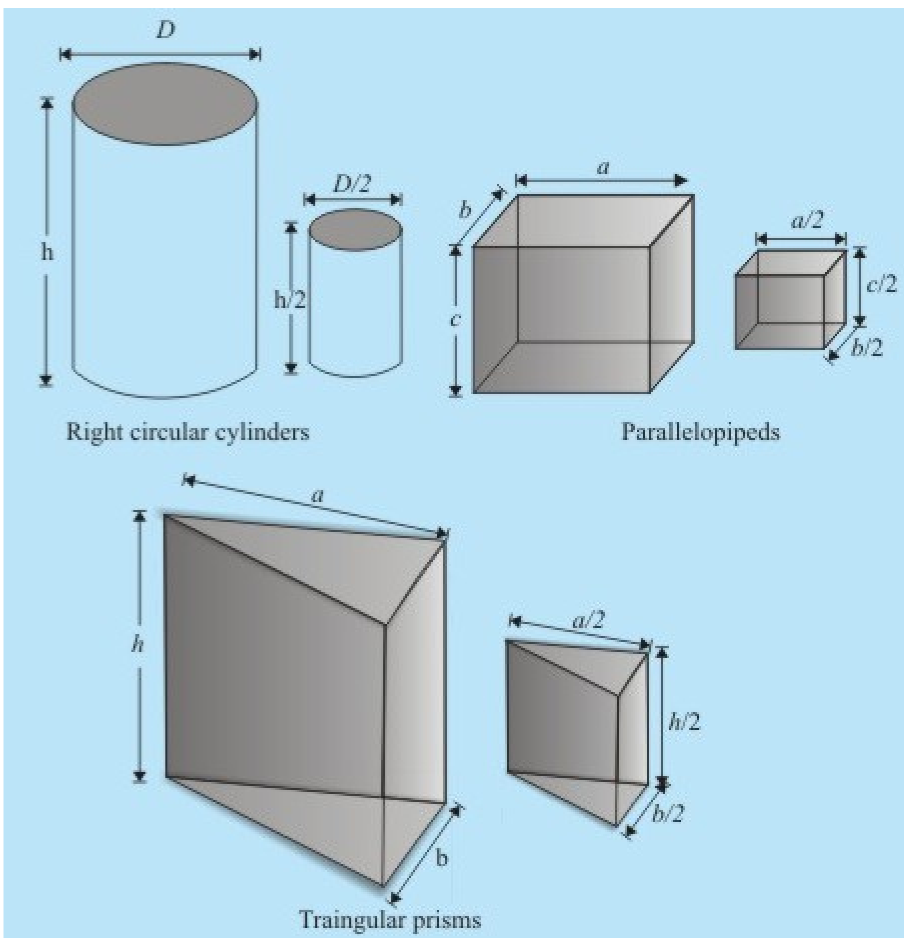


Fig 17.1 Geometrically Similar Objects

In all the above cases model ratio is 1/2

Geometric similarity is perhaps the most obvious requirement in a model system designed to correspond to a given prototype system.

A perfect geometric similarity is not always easy to attain. **Problems in achieving perfect geometric similarity** are:

- For a small model, the surface roughness might not be reduced according to the scale factor (unless the model surfaces can be made very much smoother than those of the prototype). If for any reason the scale factor is not the same throughout, a distorted model results.

- Sometimes it may so happen that to have a perfect geometric similarity within the available laboratory space, physics of the problem changes. For example, in case of large prototypes, such as rivers, the size of the model is limited by the available floor space of the laboratory; but if a very low scale factor is used in reducing both the horizontal and vertical lengths, this may result in a stream so shallow that surface tension has a considerable effect and, moreover, the flow may be laminar instead of turbulent. In this situation, a distorted model may be unavoidable (a lower scale factor ”for horizontal lengths while a relatively higher scale factor for vertical lengths. The extent to which perfect geometric similarity should be sought therefore depends on the problem being investigated, and the accuracy required from the solution.

Kinematic Similarity

Kinematic similarity refers to **similarity of motion.**

Since motions are described by distance and time, it implies **similarity of lengths (i.e., geometrical similarity**) and, in addition**, similarity of time intervals**.

If the corresponding lengths in the two systems are in a fixed ratio, the velocities of corresponding particles must be in a fixed ratio of magnitude of corresponding time intervals.

If the ratio of corresponding lengths, known as the **scale factor, is l_r** and **the ratio of corresponding time intervals is t_r**, then the magnitudes of corresponding **velocities are in the ratio l_r/t_r** and the magnitudes of corresponding **accelerations are in the ratio $l_r/t^2{}_r$**.

A well-known **example** of kinematic similarity is found in a planetarium. Here the galaxies of stars and planets in space are reproduced in accordance with a certain length scale and in simulating the motions of the planets, a fixed ratio of time intervals (and hence velocities and accelerations) is used.

When fluid motions are kinematically similar, the **patterns formed by streamlines are geometrically similar** at corresponding times.

Since the impermeable boundaries also represent streamlines, **kinematically similar flows are possible only past geometrically similar boundaries.**

Therefore, **geometric similarity is a necessary condition for the kinematic similarity** to be achieved, but not the sufficient one.

For example, geometrically similar boundaries may ensure geometrically similar streamlines in the near vicinity of the boundary but not at a distance from the boundary.

Dynamic Similarity

Dynamic similarity is the **similarity of forces** .

In dynamically similar systems, the **magnitudes of forces** at correspondingly similar points in each system are **in a fixed ratio.**

In a system involving flow of fluid, different forces due to different causes may act on a fluid element. These forces are as follows:

Viscous Force (due to viscosity)	$\vec{F}_v$
Pressure Force (due to different in pressure)	$\vec{F}_p$
Gravity Force (due to gravitational attraction)	$\vec{F}_g$
Capillary Force (due to surface tension)	$\vec{F}_c$
Compressibility Force (due to elasticity)	$\vec{F}_e$

According to Newton 's law, the resultant F_R of all these forces, will cause the acceleration of a fluid element. Hence

$$\vec{F}_R = \vec{F}_v + \vec{F}_p + \vec{F}_g + \vec{F}_c + \vec{F}_e \qquad (17.1)$$

Moreover, the **inertia force** $\vec{F}_i$ **is defined as equal and opposite to the resultant accelerating force** $\vec{F}_R$

$$\vec{F}_i = -\vec{F}_R$$

Therefore Eq. 17.1 can be expressed as

$$\vec{F}_v + \vec{F}_p + \vec{F}_g + \vec{F}_c + \vec{F}_e + \vec{F}_i = 0$$

For dynamic similarity, the magnitude ratios of these forces have to be same for both the prototype and the model. **The inertia force $\vec{F}_i$ is usually taken as the common one to describe the ratios** as (or putting in other form we equate the the non dimensionalised forces in the two systems)

$$\frac{|\vec{F}_v|}{|\vec{F}_i|}, \frac{|\vec{F}_p|}{|\vec{F}_i|}, \frac{|\vec{F}_g|}{|\vec{F}_i|}, \frac{|\vec{F}_\sigma|}{|\vec{F}_i|}, \frac{|\vec{F}_e|}{|\vec{F}_i|}$$

9.4 Applications:

The Application of Dynamic Similarity - The Dimensional Analysis

The concept:

A physical problem may be characterised by a group of dimensionless similarity parameters or variables rather than by the original dimensional variables.

This gives a clue to the reduction in the number of parameters requiring separate consideration in an experimental investigation.

For an **example,** if the Reynolds number $Re = \rho V D_h / \mu$ is considered as the independent variable, in case of a flow of fluid through a closed duct of hydraulic diameter D_h, then a change in Re may be caused through a change in flow velocity V only. Thus a range of Re can be covered simply by the variation in V without varying other independent dimensional variables ρ, D_h and μ.

In fact, the variation in the Reynolds number physically implies the variation in any of the dimensional parameters defining it, though the change in Re, may be obtained through the variation in anyone parameter, say the velocity V.

A number of such **dimensionless parameters** in relation to dynamic similarity are shown in Table 5.1. Sometimes it becomes diffcult to derive these parameters straight forward from an estimation of the representative order of magnitudes of the forces involved. An alternative **method of determining these dimensionless parameters by a mathematical technique is known as dimensional analysis** .

The Technique:

The requirement of dimensional homogeneity imposes conditions on the quantities involved in a physical problem, and these restrictions, placed in the form of an algebraic function by the requirement of dimensional homogeneity, play the central role in dimensional analysis.

There are two existing approaches;

- one due to Buckingham known as **Buckingham's pi theorem**
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Chapter 10
Elements of Compressible Flow

10.1 Compressible Flow Properties:

Introduction

- **Compressible flow** is often called as **variable density flow**. For the flow of all liquids and for the flow of gases under certain conditions, the density changes are so small that assumption of constant density remains valid.

- Let us consider a small element of fluid of volume $\forall$. The pressure exerted on the element by the neighbouring fluid is p . If the pressure is now increased by an amount dp , the volume of the element will correspondingly be reduced by the amount d $\forall$.The **compressibility** of the fluid K is thus defined as

$$K = \frac{1}{\rho}\cdot\frac{d\rho}{dp} \qquad (38.1)$$

However, when a gas is compressed, its temperature increases. Therefore, the above mentioned definition of compressibility is not complete unless temperature condition is specified. When the temperature is maintained at a constant level, the **isothermal compressibility** is defined as

$$K_T = -\frac{1}{\forall}\left(\frac{d\forall}{dp}\right)_T \qquad (38.2)$$

- Compressibility is a property of fluids. Liquids have very low value of compressibility (for ex. compressibility of water is 5 ´ 10^{-10} m^2/N at 1 atm under isothermal condition), while gases have very high compressibility (for ex. compressibility of air is 10^{-5} m^2/N at 1 atm under isothermal condition).

- If the fluid element is considered to have unit mass and v is the specific volume (volume per unit mass) , the density is $\rho = \frac{1}{v}$. In terms of density; Eq. (38.1) becomes

$$\rho = \frac{1}{v} \Rightarrow d\rho = -\frac{1}{v^2} dv \quad (38.3)$$

$$K = \frac{1}{\rho} \cdot \frac{d\rho}{dp}$$

We can say that from Eqn (38.1) for a change in pressure, dp, the change in density is

$$d\rho = \rho \cdot K \cdot dp \quad (38.4)$$

- If we also consider the fluid motion, we shall appreciate that the flows are initiated and maintained by changes in pressure on the fluid. It is also known that high pressure gradient is responsible for high speed flow. However**, for a given pressure gradient dp , the change in density of a liquid will be much smaller than the change in density of a gas** (as seen in Eq. (38.4)).

So, for flow of gases, moderate to high pressure gradients lead to substantial changes in the density. Due to such pressure gradients, gases flow with high velocity. **Such flows, where ρ is a variable, are known as compressible flows.**

- Recapitulating Chapter 1, we can say that the proper criterion for a nearly incompressible flow is a small Mach number,

$$Ma = \frac{V}{a} << 1 \quad (38.5)$$

- where V is the flow velocity and a is the speed of sound in the fluid. For small Mach number, changes in fluid density are small everywhere in the flow field.

- In this chapter we shall treat compressible flows which have Mach numbers greater than 0.3 and exhibit appreciable density changes. The **Mach number is the most important parameter in compressible flow analysis**. Aerodynamicists make a distinction between different regions of Mach number.

  Categories of flow for external aerodynamics.

  - Ma < 0.3: **incompressible flow**; change in density is negligible.
  - 0.3< Ma < 0.8: **subsonic flow**; density changes are significant but shock waves do not appear.
  - 0.8< Ma < 1.2: **transonic flow**; shock waves appear and divide the subsonic and supersonic regions of the flow. Transonic flow is characterized by mixed regions of locally subsonic and supersonic flow
  - 1.2 < Ma < 3.0: **supersonic flow**; flow field everywhere is above acoustic speed. Shock waves appear and across the shock wave, the streamline changes direction discontinuously.
  - 3.0< Ma : **hypersonic flow**; where the temperature, pressure and density of the flow increase almost explosively across the shock wave.

- **For internal flow**, it is to be studied **whether the flow is subsonic (** Ma < 1) **or supersonic** (Ma > 1)**.** The effect of change in area on velocity changes in subsonic and supersonic regime is of considerable interest. By and large, in this chapter we shall mostly focus our attention to internal flows.

Perfect Gas

- A perfect gas is one in which **intermolecular forces are neglected**. The equation of state for a perfect gas can be derived from kinetic theory. It was synthesized from laboratory experiments by Robert Boyle, Jacques Charles, Joseph Gay-Lussac and John Dalton. For a perfect gas, it can be written

$$p\forall = MRT \tag{38.6}$$

- where p is pressure (N/m^2), $\forall$ is the volume of the system (m^3), M is the mass of the system (kg), R is the characteristic gas constant (J/kg K) and T is the temperature (K). This equation of state can be written as

$$pv = RT \tag{38.7}$$

-

where v is the specific volume (m^3/kg). Also,

$$p = \rho R T \qquad (38.8)$$

where ρ is the density (kg/m^3).

- In another approach, which is particularly useful in chemically reacting systems, the equation of state is written as

$$p\forall = N\Re T \qquad (38.9)$$

- where N is the number of moles in the system, and $\Re$ is the universal gas constant which is same for all gases

  .

- Recall that a mole of a substance is that amount which contains a mass equal to the molecular weight of the gas and which is identified with the particular system of units being used. For example, in case of oxygen (O_2), 1 kilogram-mole (or kg. mol) has a mass of 32 kg. Because the masses of different molecules are in the same ratio as their molecular weights; 1 mol of different gases always contains the same number of molecules, i.e. 1 kg-mol always contains 6.02×10^{26} molecules, independent of the species of the gas. Dividing Eq. (38.9) by the number of moles of the system yields

$$p\forall^{1} = \Re T \qquad (38.10)$$

- $\forall^{1}$: Vol. per unit mole
- If Eq. (38.9) is divided by the mass of the system, we can write

$$pv = \eta \Re T \qquad (38.11)$$

- where v is the specific volume as before and η is the mole-mass ratio (kg-mol/kg). Also, Eq. (38.9) can be divided by system volume, which results in

$$p = C\Re T \qquad (38.12)$$

- where C is the concentration (kg - mol/m^3)

- The equation of state can also be expressed in terms of particles. If N_A is the number of molecules in a mole (**Avogadro** constant, which for a kilogram-mole is 6.02 $\times 10^{26}$ particles), from Eq. (38.12) we obtain

$$p = (N_A C)\left(\frac{\Re}{N_A}\right)T \qquad (38.13)$$

In the above equation, $N_A C$ is the number density, i.e. number of particles per unit volume and $\frac{\Re}{N_A}$ is the gas constant per particle, which is nothing but Boltzmann constant.

Finally, Eq. (38.13) can be written as

$$p = n\kappa T \qquad (38.14)$$

where n**: number density**
κ**: Boltzmann constant.**

- It is interesting to note that there exist a variety of gas constants whose use depends on the equation in consideration.

1.Universal gas constant- When the equation deals with moles, it is in use. It is same for all the gases.

$\Re$**= 8314 J/(Kg-mol-K)**

2.Characteristic gas constant- When the equation deals with mass, the characteristic gas constant (R) is used. It is a gas constant per unit mass and it is different for different gases. As such $R = \Re / M$, where M is the molecular weight. For air at standard conditions,

R = 287 J/(kg-K)

3.Boltzmann constant- When the equation deals with molecules, Boltzmann constant is used. It is a gas constant per unit molecule .

$\kappa = 1.38 \text{ X } 10^{-23} \text{J / K}$

Application of the perfect gas theory

a. It has been experimentally determined that at low pressures (1 atm or less) and at high temperature (273 K and above), the value of $\frac{pv}{RT}$ (the well known compressibility z, of a gas) for most pure gases differs from unity by a quanity less than one percent (the well known compressibility z, of a gas).
b. Also, that at very low temperatures and high pressures the molecules are densely packed. Under such circumstances, the gas is defined as real gas and the perfect gas equation of state is replaced by the famous **Van-der-Waals equation** which is

$$\left(p+\frac{a}{v^2}\right)(v-b)=RT \qquad (38.15)$$

where a and b are constants and depend on the type of the gas.

In conclusion, it can be said that for a wide range of applications related to compressible flows, the temperatures and pressures are such that the equation of state for the perfect gas can be applied with high degree of confidence.

Internal Energy and Enthalpy

- Microscopic view of a gas is a collection of particles in random motion. Energy of a particle consists of **translational energy, rotational energy, vibrational energy** and **specific electronic energy**. All these energies summed over all the particles of the gas, form the specific internal energy, e , of the gas.
- Imagine a gas in thermodynamic equilibrium,i.e., gradients in velocity,

pressure, temperature and chemical concentrations do not exist.

Then the enthalpy, h , is defined as $h = e + p\upsilon$, where υ is the specific volume.

$$e = e(T, \forall)$$
$$h = h(T, p) \qquad (38.16)$$

If the gas is not chemically reacting and the intermolecular forces are neglected, the system can be called as a **thermally perfect gas**, where internal energy and enthalpy are functions of temperature only. One can write

$$e = e(T)$$
$$h = h(T)$$
$$de = c_v dT \qquad (38.17)$$
$$dh = c_p dT$$

For a calorically perfect gas,

$$e = c_v T$$
$$h = c_p T \qquad (38.18)$$

Please note that in most of the compressible flow applications, the pressure and temperatures are such that the gas can be considered as calorically perfect.

- For calorically perfect gases, we assume constant specific heats and write

$$c_p - c_v = R \qquad (38.19)$$

- The specific heats at constant pressure and constant volume are defined as

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p \qquad c_v = \left(\frac{\partial e}{\partial T}\right)_\upsilon \qquad (38.20)$$

Equation (38.19), can be rewritten as

$$1-\frac{c_v}{c_p}=\frac{R}{c_p} \qquad (38.21)$$

Also $\frac{c_p}{c_v}=\gamma$. So we can rewrite Eq. (38.21) as

$$1-\frac{1}{\gamma}=\frac{R}{c_p} \qquad (38.22)$$

$$c_p=\frac{\gamma R}{\gamma-1}$$

In a similar way, from Eq. (38.19) we can write

$$c_v=\frac{R}{\gamma-1} \qquad (38.23)$$

10.2 Total Enthalpy:

Stagnation and Sonic Properties

- The stagnation properties at a point are defined as those which are to be obtained if the local flow were imagined to cease to zero velocity isentropically. As we will see in the later part of the text, stagnation values are useful reference conditions in a compressible flow.

Let us denote stagnation properties by subscript zero. Suppose the properties of a flow (such as T, p , ρ etc.) are known at a point, the stangation enthalpy is, thus, defined as

$$h_0=h+\frac{1}{2}V^2$$

where h is flow enthalpy and V is flow velocity.

- For a perfect gas , this yields,

$$c_pT_0=c_pT+\frac{1}{2}V^2 \qquad (40.1)$$

which defines the **Stagnation Temperature** (T_0)

Now, $\dfrac{T_0}{T}$ can be expressed as

$$\frac{T_0}{T} = 1 + \frac{V^2}{2c_p T} = 1 + \frac{\gamma - 1}{2} \cdot \frac{V^2}{\gamma RT}$$

Since,

$$c_p - \frac{c_p}{\gamma} = R$$

$$c_p = \frac{\gamma R}{\gamma - 1}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} Ma^2 \qquad \left(Ma = \frac{V}{a} = \frac{V}{\sqrt{\gamma RT}} \right) \tag{40.2}$$

If we know the local temperature (**T**) and Mach number **(Ma)** , we can find out the stagnation temperature $\mathbf{T_0}$.

Consequently, **isentropic(adiabatic) relations** can be used to obtain **stagnation pressure** and **stagnation density** as

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma - 1}} = \left[1 + \frac{\gamma - 1}{2} Ma^2\right]^{\frac{\gamma}{\gamma - 1}} \tag{40.3}$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma - 1}} = \left[1 + \frac{\gamma - 1}{2} Ma^2\right]^{\frac{1}{\gamma - 1}} \tag{40.4}$$

Values of T_0/T , p_0/p and ρ_0/ρ as a function of Mach number can be generated using the above relationships and the tabulated results are known as **Isentropic Table** . Interested readers are suggested to refer the following books

<u>10.3 Total Temeperature:</u> Refer Section 10.2

10.4 Temperature and Pressure ratio as function of Mach number:

Refer Section 10.2

10.5 Mass Flow Parameter:

Isentropic Flow in a Converging Nozzle

Consider the mass flow rate of an ideal gas through a converging nozzle. If the flow is isentropic, we can write

$$\dot{m} = \rho A V$$

where V is flow velocity, A is area, ρ is the density of the field.

This can equivalently be written as-

$$\frac{\dot{m}}{A} = \frac{p}{RT}\cdot a \cdot Ma$$

$$\frac{\dot{m}}{A} = \frac{p}{RT}\cdot \sqrt{\gamma RT}\cdot Ma$$

$$\frac{\dot{m}}{A} = \frac{p}{\sqrt{T}}\cdot \sqrt{\frac{\gamma}{R}}\cdot Ma$$

$$\frac{\dot{m}}{A} = \frac{p}{p_0}\cdot p_0 \cdot \sqrt{\frac{T_0}{T}}\cdot \sqrt{\frac{1}{T_0}}\cdot \sqrt{\frac{\gamma}{R}}\cdot Ma$$

$$\frac{\dot{m}}{A} = \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{T_0}{T}\right)^{\frac{1}{2}}\frac{p_0}{\sqrt{T_0}}\cdot \sqrt{\frac{1}{T_0}}\cdot Ma$$

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}}\cdot \frac{p_0 Ma}{\sqrt{T_0}}\cdot \left(\frac{T_0}{T}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0 Ma}{\sqrt{T_0}} \cdot \frac{1}{\left[1+\frac{\gamma-1}{2}Ma^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}} \qquad (40.12)$$

In the expression (40.12), p_0, T_0, γ and R are constant

- The discharge per unit area $\frac{\dot{m}}{A}$ is a function of Ma only. There exists a particular value of Ma for which it is maximum. Differentiating with respect to Ma and equating it to zero, we get

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0}{\sqrt{T_0}} \cdot \frac{1}{\left[1+\frac{\gamma+1}{2}Ma^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

$$+\sqrt{\frac{\gamma}{R}} \cdot \frac{p_0 Ma}{\sqrt{T_0}}\left[-\frac{\gamma+1}{2(\gamma-1)}\left\{1+\frac{\gamma-1}{2}Ma^2\right\}\left(\frac{\gamma+1}{2(\gamma-1)}-1\right)\left\{\frac{\gamma-1}{2}2Ma\right\}\right]=0$$

$$\Rightarrow 1-\frac{Ma^2(\gamma+1)}{2\left\{1+\frac{\gamma-1}{2}Ma^2\right\}}=0$$

$$\Rightarrow Ma^2(\gamma+1)=2+(\gamma-1)Ma^2$$

$$\Rightarrow Ma=1$$

Hence, discharge is maximum when Ma = 1.

10.6 Isentropic area ratio A/A*:

- We know that $V = aMa = \sqrt{\gamma RT}\,Ma$. By logarithmic differentiation, we get

$$\frac{dV}{V} = \frac{dMa}{Ma} + \frac{1}{2}\frac{dT}{T} \tag{40.13}$$

We also know that

$$\frac{T}{T_0} = \left[1 + \frac{\gamma - 1}{2}Ma^2\right]^{-1}$$

By logarithmic differentiation, we get

$$\frac{dT}{T} = -\frac{(\gamma - 1)Ma^2}{1 + \frac{\gamma - 1}{2}Ma^2}\cdot\frac{dMa}{Ma} \tag{40.14}$$

From Eqs(40.13) and (40.14) , we get

$$\frac{dV}{V} = \frac{dMa}{Ma}\left[1 - \frac{\frac{\gamma - 1}{2}Ma^2}{1 + \frac{\gamma - 1}{2}Ma^2}\right]$$

$$\Rightarrow \frac{dV}{V} = \frac{dMa}{Ma}\left[\frac{1}{1 + \frac{\gamma - 1}{2}Ma^2}\right] \tag{40.15}$$

From Eqs (40.11) and (40.15), we get

$$\frac{dA}{A}\frac{1}{\left(Ma^2 - 1\right)} = \frac{1}{1 + \frac{\gamma - 1}{2}Ma^2}\frac{dMa}{Ma}$$

$$\frac{dA}{A} = \frac{\left(Ma^2 - 1\right)}{1 + \frac{\gamma - 1}{2}Ma^2}\frac{dMa}{Ma} \tag{40.16}$$

By substituting Ma=1 in Eq. (40.16), we get dA = 0 or A = constant.

- Ma=1 can occur only at the throat and nowhere else, and this happens only when the discharge is maximum. **When Ma = 1 , the discharge is maximum and the nozzle is said to be choked.**

  The properties at the throat are termed as critical properties which are already expressed through Eq. (40.6a), (40.6b) and (40.6c). By substituting Ma = 1 in Eq. (40.12), we get

$$\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0}{\sqrt{T_0}} \cdot \frac{1}{\left[\frac{(\gamma+1)}{2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} \qquad (40.17)$$

(as we have earlier designated critical or sonic conditions by a superscript asterisk). Dividing Eq. (40.17) by Eq. (40.12) we obtain

$$\frac{A}{A^*} = \frac{1}{Ma}\left[\left\{\frac{2}{\gamma+1}\right\}\left\{1+\frac{(\gamma-1)}{2}Ma^2\right\}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \qquad (40.18)$$

From Eq. (40.18) we see that a choice of Ma gives a unique value of A/A* . The following figure shows variation of A / A * with Ma (Fig 40.6). Note that the curve is double valued; that is, for a given value of A/A* (other than unity), there are two possible values of Mach number. **This signifies the fact that the supersonic nozzle is diverging.**

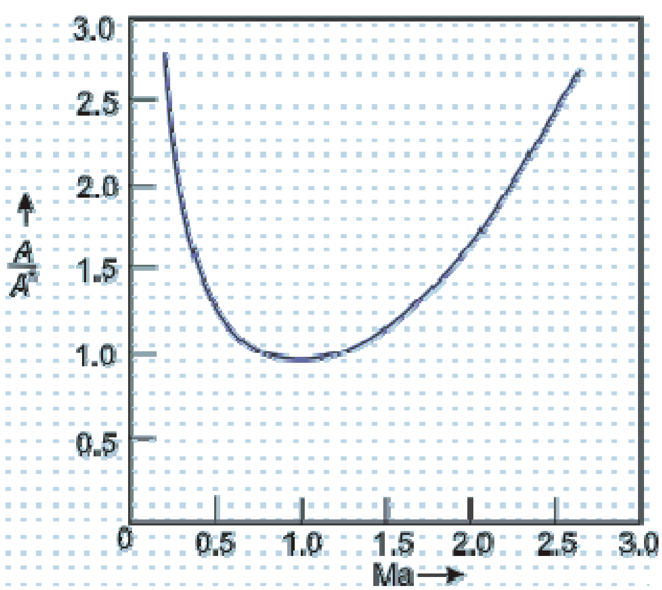

Fig 40.6: Variation of A/A* with Ma in isentropic flow for γ = 1.4

10.7 Velocity Area Variation:

Effect of Area Variation on Flow Properties in Isentropic Flow

In considering the effect of area variation on flow properties in isentropic flow, we shall determine the effect on the velocity V and the pressure p .

From Eq . (39.11), we can write

$$\frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) = 0$$

$$dp = -\rho V dV$$

Dividing by ρV^2, we obtain

$$\frac{dp}{\rho V^2} = -\frac{dV}{V} \qquad (40.8)$$

A convenient differential form of the continuity equation can be obtained from Eq. (39.6) as

$$\frac{dA}{A} = -\frac{dV}{V} - \frac{d\rho}{\rho}$$

Substituting from Eq. (40.8),

$$\frac{dA}{A} = \frac{dp}{\rho V^2} - \frac{d\rho}{\rho}$$

$$\frac{dA}{A} = \frac{dp}{\rho V^2}\left[1 - \frac{V^2}{dp/d\rho}\right] \qquad (40.9)$$

Invoking the relation (39.3b) for isentropic process in Eq. (40.9), we get

$$\frac{dA}{A} = \frac{dp}{\rho V^2}\left[1 - \frac{V^2}{a^2}\right] = \frac{dp}{\rho V^2}\left[1 - Ma^2\right] \qquad (40.10)$$

- From Eq.(40.10), we see that for Ma<1 an area change causes a pressure change of the same sign, i.e. positive dA means positive dp for Ma<1 . For Ma>1 , an area change causes a pressure change of opposite sign.

- Again, substituting from Eq. (40.8) into Eq. (40.10), we obtain

$$\frac{dA}{A} = -\frac{dV}{V}\left[1 - Ma^2\right] \tag{40.11}$$

From Eq. (40.11) we see that Ma<1 an area change causes a velocity change of opposite sign, i.e. positive dA means negative dV for Ma<1 . For Ma>1 an area change causes a velocity change of same sign.

These results can be summarized in fig 40.2. Equations (40.10) and (40.11) lead to the following important conclusions about compressible flows:

1. At subsonic speeds**(**Ma<1**)** a decrease in area increases the speed of flow. A subsonic nozzle should have a convergent profile and a subsonic diffuser should possess a divergent profile. The flow behaviour in the regime of Ma<1 is therefore qualitatively the same as in incompressible flows.
2. In supersonic flows **(**Ma>1**)** the effect of area changes are different. According to Eq. (40.11), a supersonic nozzle must be built with an increasing area in the flow direction. A supersonic diffuser must be a converging channel. Divergent nozzles are used to produce supersonic flow in missiles and launch vehicles.

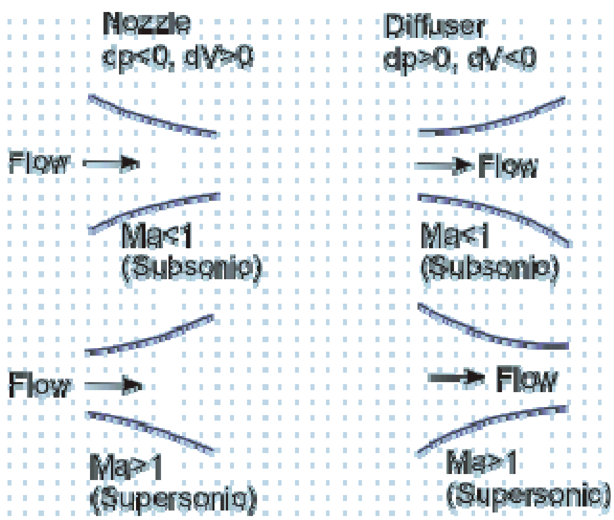

Fig 40.2 Shapes of nozzles and diffusers in subsonic and supersonic regimes

10.8 2D Small Amplitude wave propagation:

Speed of Sound

- The so-called sound speed is the **rate of propogation of a pressure pulse** of infinitesimal strength through a still fluid. It is a **thermodynamic property** of a fluid.

- A pressure pulse in an incompressible flow behaves like that in a rigid body. A displaced particle displaces all the particles in the medium. In a compressible fluid, on the other hand, displaced mass compresses and increases the density of neighbouring mass which in turn increases density of the adjoining mass and so on. Thus, a disturbance in the form of an elastic wave or a pressure wave travels through the medium. If the amplitude and theerfore the strength of the elastic wave is infinitesimal, it is termed as acoustic wave or sound wave.

- Figure 39.1(a) shows an infinitesimal pressure pulse propagating at a speed " a " towards still fluid ($V = 0$) at the left. The fluid properties ahead of the wave are p,T and ρ, while the properties behind the wave are p+dp, T+dT and $\rho + d\rho$. The fluid velocity dV is directed toward the left following wave but much slower.

- In order to make the analysis steady, we superimpose a velocity " a " directed towards right, on the entire system (Fig. 39.1(b)). The wave is now stationary and the fluid appears to have velocity " a " on the left and (a - dV) on the right. The flow in Fig. 39.1 (b) is now steady and one dimensional across the wave. Consider an area A on the wave front. A mass balance gives

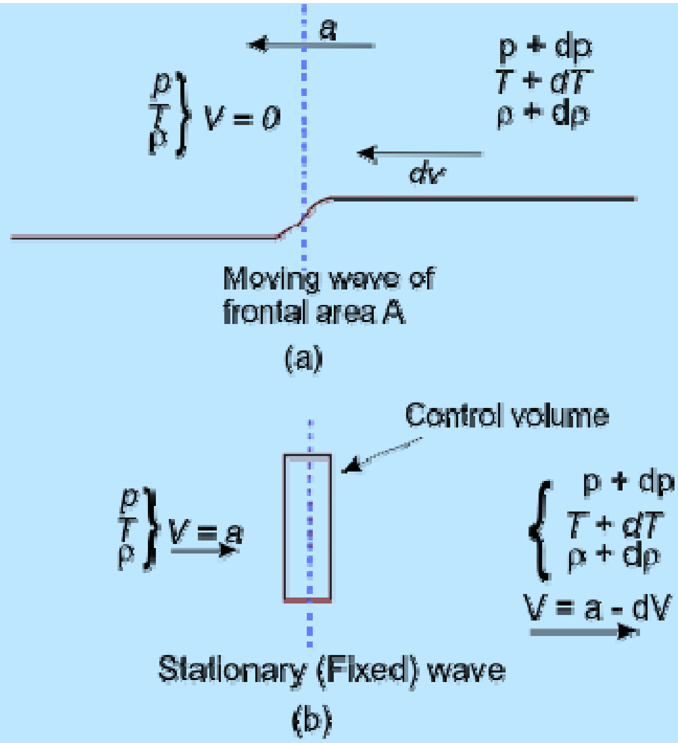

Fig 39.1: Propagation of a sound wave
(a) Wave Propagating into still Fluid (b) Stationary Wave

$$\rho A a = (\rho + d\rho) A (a - dV)$$

$$dV = a\left[\frac{d\rho}{\rho + d\rho}\right] \qquad (39.1)$$

This shows that

(a) $dv > 0$ if dρ is positive.

(b) A compression wave leaves behind a fluid moving in the direction of the wave (Fig. 39.1(a)).

(c) Equation (39.1) also signifies that the fluid velocity on the right is much smaller than the wave speed " a ". Within the framework of infinitesimal strength of the wave (sound wave), this " a " itself is very small.

- Applying the momentum balance on the same control volume in Fig. 39.1 (b). It says that the net force in the x direction on the control volume equals the rate of outflow of x momentum minus the rate of inflow of x momentum. In symbolic form, this yields

$$pA - (p + dp)A = (A\rho a)(a - dV) - (A\rho a)(a)$$

In the above expression, $A\rho a$ is the mass flow rate. The first term on the right hand side represents the rate of outflow of x-momentum and the second term represents the rate of inflow of x momentum.

- Simplifying the momentum equation, we get

$$dp = \rho a dV \qquad (39.2)$$

If the wave strength is very small, the pressure change is small.

Combining Eqs (39.1) and (39.2), we get

$$a^2 = \frac{dp}{d\rho}\left(1 + \frac{d\rho}{\rho}\right) \qquad (39.3a)$$

The larger the strength $d\rho/\rho$ of the wave ,the faster the wave speed; i.e., powerful explosion waves move much faster than sound waves.In the limit of infinitesimally small strength, $d\rho \rightarrow 0$ we can write

$$a^2 = \frac{dp}{d\rho} \qquad (39.3b)$$

Note that

(a) In the limit of infinitesimally strength of sound wave, there are no velocity gradients on either side of the wave. Therefore, the frictional effects (irreversible) are confined to the interior of the wave.

(b) Moreover, the entire process of sound wave propagation is adiabatic because there is no temperature gradient except inside the wave itself.

(c) So, for sound waves, we can see that the process is reversible adiabatic or **isentropic**.

So the correct expression for the sound speed is

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \qquad (39.4)$$

For a perfect gas, by using of $\frac{p}{\rho^{\gamma}} = const$, and $p = \rho RT$, we deduce the speed of sound as

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT} \qquad (39.5)$$

For air at sea-level and at a temperature of 15^0C, a=340 m/s

10.10 Description of Flow regime:

Refer Section 10.1

10.11 Introduction to Oblique and Normal Shocks:

Normal Shocks

- Shock waves are highly localized irreversibilities in the flow .
- Within the distance of a mean free path, the flow passes from a supersonic to a subsonic state, the velocity decreases suddenly and the pressure rises sharply. A shock is said to have occurred if there is an abrupt reduction of velocity in the downstream in course of a supersonic flow in a passage or around a body.
- **Normal shocks** are substantially **perpendicular** to the flow and **oblique shocks** are inclined at **any angle**.
- Shock formation is possible for confined flows as well as for external flows.
- Normal shock and oblique shock may mutually interact to make another shock pattern.

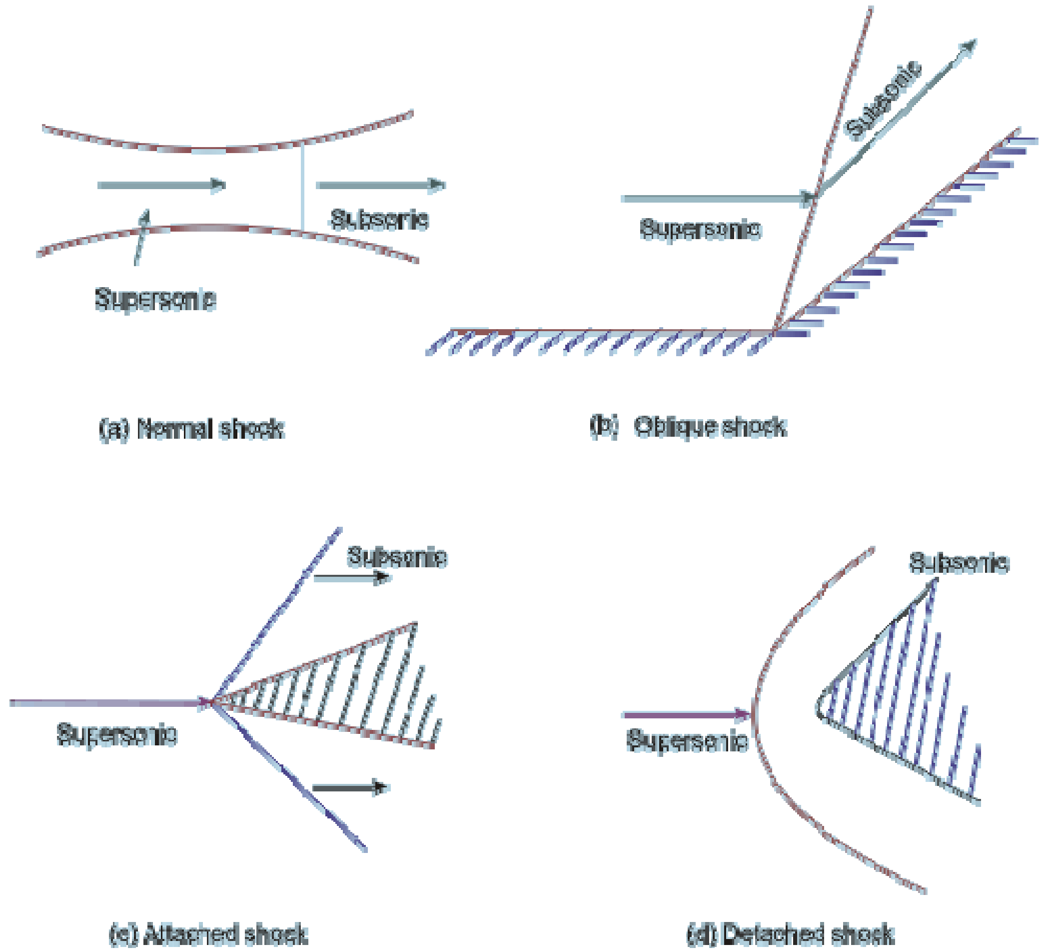

Fig 41.1 Different type of Shocks

Figure below shows a control surface that includes a normal shock.

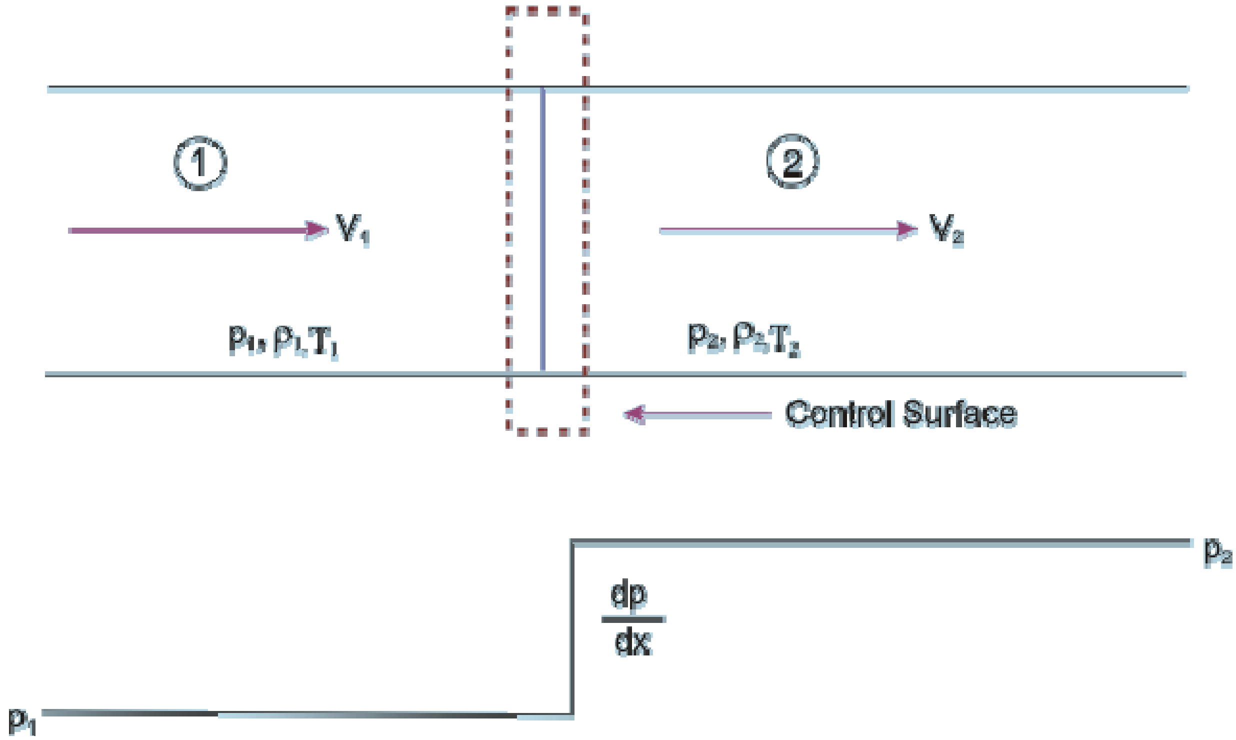

Fig 41.2 One Dimensional Normal Shock

- The fluid is assumed to be in thermodynamic equilibrium upstream and downstream of the shock, the properties of which are designated by the subscripts 1 and 2, respectively. (Fig 41.2).

  Continuity equation can be written as

$$\frac{\dot{m}}{A} = \rho_1 V_1 = \rho_2 V_2 = G \tag{41.1}$$

where G is the mass velocity kg/ m^2 s, and $\dot{m}$ is mass flow rate

From momentum equation, we can write

$$p_1 - p_2 = \frac{\dot{m}}{A}(V_2 - V_1) = \rho_2 V_2^2 - \rho_1 V_1^2 \tag{41.2a}$$

$$\Rightarrow p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \tag{41.2b}$$

$$\Rightarrow \quad F_1 = F_2$$

where p + ρV^2 is termed as **Impulse Function** .

The energy equation is written as

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_{01} = h_{02} = h_0 \tag{41.3}$$

where h_0 is **stagnation enthalpy**.

From the second law of thermodynamics, we know

$$s_2 - s_1 \geq 0$$

To calculate the entropy change, we have

$$Tds = dh - \forall dp$$

For an ideal gas

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

For an ideal gas the equation of state can be written as

$$p = \rho RT \tag{41.4}$$

For constant specific heat, the above equation can be integrated to give

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} \tag{41.5}$$

Equations (41.1), (41.2a), (41.3), (41.4) and (41.5) are the governing equations for the flow of an ideal gas through normal shock.

If all the properties at state 1 (upstream of the shock) are known, then we have six unknowns $T_2, p_2, \rho_2, V_2, h_2, s_2$ in these five equations.

We know relationship between h and T [Eq. (38.17)] for an ideal gas, $dh = c_p dT$. For an ideal gas with constant specific heats,

$$\Delta h = h_2 - h_1 = c_p(T_2 - T_1) \tag{41.6}$$

Thus, we have the situation of six equations and six unknowns.

- If all the conditions at state "1"(immediately upstream of the shock) are known, how many possible states 2 (immediate downstream of the shock) are there? **The mathematical answer indicates that there is a unique state 2 for a given state 1.**

Calculation of Flow Properties Across a Normal Shock

- The easiest way to analyze a normal shock is to consider a control surface around the wave as shown in Fig. 41.2. **The continuity equation** (41.1), **the momentum equation** (41.2) and the **energy equation** (41.3) have already been discussed earlier. The energy equation can be simplified for an ideal gas as

$$T_{01} = T_{02} \qquad (40.9)$$

- By making use of the equation for the speed of sound eq. (39.5) and the equation of state for ideal gas eq. (38.8), the continuity equation can be rewritten to include the influence of Mach number as:

$$\frac{p_1}{RT_1} Ma_1 \sqrt{\gamma R T_1} = \frac{p_2}{RT_2} Ma_2 \sqrt{\gamma R T_2} \qquad (40.10)$$

Introducing the Mach number in momentum equation, we have

$$\rho_2 V_2^2 - \rho_1 V_1^2 = p_1 - p_2$$

$$p_1 + \frac{p_1}{RT_1} V_1^2 = p_2 + \frac{p_2}{RT_2} V_2^2$$

Therefore ,

$$p_1\left(1 + \gamma Ma_1^2\right) = p_2\left(1 + \gamma Ma_2^2\right) \qquad (40.11)$$

Rearranging this equation for the static pressure ratio across the shock wave, we get

$$\frac{p_2}{p_1} = \frac{\left(1 + \gamma\, Ma_1^2\right)}{\left(1 + \gamma\, Ma_2^2\right)} \qquad (40.12)$$

- As already seen, the Mach number of a normal shock wave is always greater than unity in the upstream and less than unity in the downstream, the static pressure always increases across the shock wave.

- The energy equation can be written in terms of the temperature and Mach number using the stagnation temperature relationship (40.9) as

$$\frac{T_2}{T_1} = \frac{\{1+(\gamma-1)/2\}Ma_1^2}{\{1+(\gamma-1)/2\}Ma_2^2} \tag{40.13}$$

Substituting Eqs (40.12) and (40.13) into Eq. (40.10) yields the following relationship for the Mach numbers upstream and downstream of a normal shock wave:

$$\frac{Ma_1}{1+\gamma Ma_1^2}\left[1+\frac{\gamma-1}{2}Ma_1^2\right]^{\frac{1}{2}} = \frac{Ma_2}{1+\gamma Ma_2^2}\left[1+\frac{\gamma-1}{2}Ma_2^2\right]^{\frac{1}{2}} \tag{40.14}$$

Then, solving this equation for Ma_2 as a function of Ma_1 we obtain two solutions. One solution is trivial $Ma_2 = Ma_1$, which signifies no shock across the control volume. The other solution is

$$Ma_2^2 = \frac{(\gamma-1)Ma_1^2+2}{2\gamma Ma_1^2-(\gamma-1)} \tag{40.15}$$

$Ma_1 = 1$ in Eq. (40.15) results in $Ma_2 = 1$

Equations (40.12) and (40.13) also show that there would be no pressure or temperature increase across the shock. In fact, the shock wave corresponding to $Ma_1 - 1$ is the sound wave across which, by definition, pressure and temperature changes are infinitesimal. Therefore, it can be said that the sound wave represents a degenerated normal shock wave. The pressure, temperature and Mach number (Ma_2) behind a normal shock as a function of the Mach number Ma_1, in front of the shock for the perfect gas can be represented in a tabular form (known as Normal Shock Table). The interested readers may refer to Spurk[1] and Muralidhar and Biswas[2].

Oblique Shock

- The discontinuities in supersonic flows do not always exist as normal to the flow direction. There are oblique shocks which are inclined with respect to the flow direction. Refer to the shock structure on an obstacle, as depicted qualitatively in Fig.41.6.
- The segment of the shock immediately in front of the body behaves like a normal shock.

- Oblique shock can be observed in following cases-

1. **Oblique shock formed as a consequence of the bending of the shock in the free-stream direction** (shown in Fig.41.6)
2. **In a supersonic flow through a duct, viscous effects cause the shock to be oblique near the walls, the shock being normal only in the core region.**
3. **The shock is also oblique when a supersonic flow is made to change direction near a sharp corner**

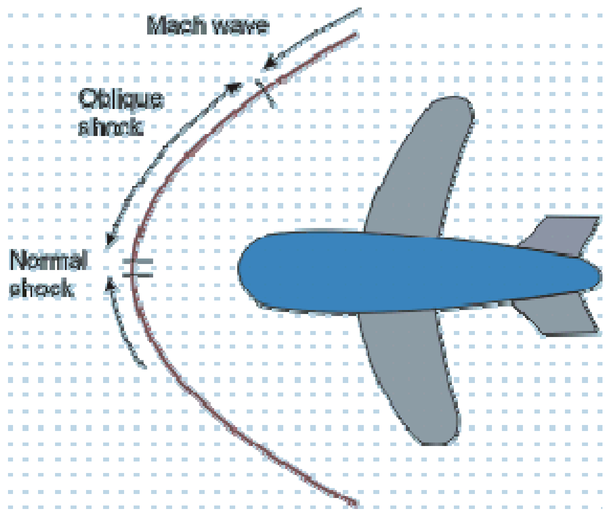

Fig 41.6 Normal and oblique Shock in front of an Obstacle

- The relationships derived earlier for the normal shock are valid for the velocity components normal to the oblique shock. The oblique shock continues to bend in the downstream direction until the Mach number of the velocity component normal to the wave is unity. At that instant, **the oblique shock degenerates into a so called Mach wave across which changes in flow properties are infinitesimal.**

- Let us now consider a two-dimensional oblique shock as shown in Fig.41.7 below

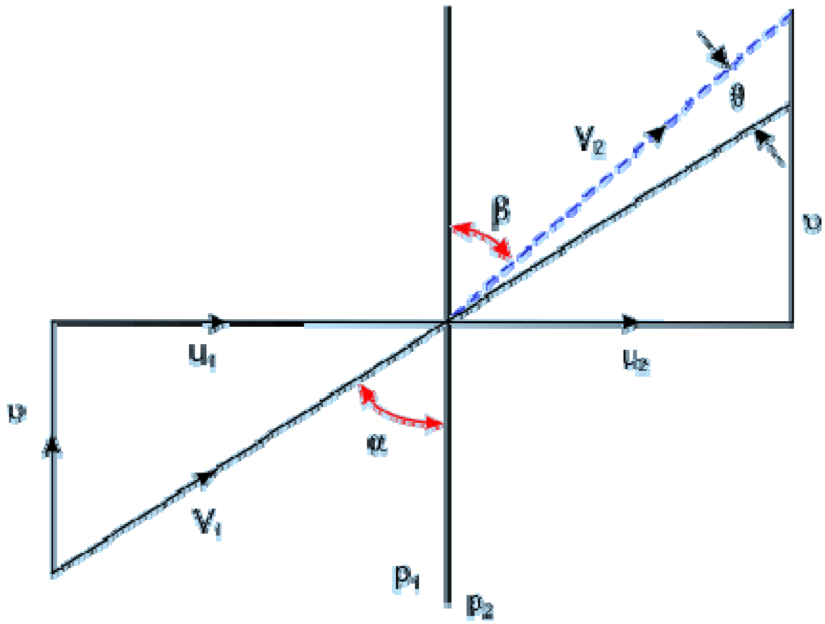

Fig 41.7 Two dimensional Oblique Shock

For analyzing flow through such a shock, it may be considered as a normal shock on which a velocity υ(parallel to the shock) is superimposed. The change across shock front is determined in the same way as for the normal shock. The equations for mass, momentum and energy conservation , respectively, are

$$\rho_1 u_1 = \rho_2 u_2 \tag{41.16}$$

$$\rho_1 u_1 (u_1 - u_2) = p_2 - p_1 \tag{41.17}$$

$$h_{01} = h_{02}$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$\frac{\gamma}{\gamma - 1} \cdot \frac{p_1}{\rho_1} + \frac{u_1^2}{2} = \frac{\gamma}{\gamma - 1} \cdot \frac{p_2}{\rho_2} + \frac{u_2^2}{2} \tag{41.18}$$

These equations are analogous to corresponding equations for normal shock. In addition to these, we have

$$\frac{u_1}{a_1} = Ma_1 \sin \alpha \quad \text{and} \quad \frac{u_2}{a_2} = Ma_2 \sin \beta$$

Modifying normal shock relations by writing $Ma_1 \sin \alpha$ and $Ma_2 \sin \beta$ in place of Ma_1 and Ma_2, we obtain

$$\frac{p_2}{p_1} = \frac{2\gamma Ma_1^2 \sin \alpha - \gamma + 1}{\gamma + 1} \qquad (41.19)$$

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{\tan \alpha}{\tan \beta} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1) + Ma_1^2 \sin^2 \alpha} \qquad (41.20)$$

$$Ma_2^2 \sin^2 \beta = \frac{2 + (\gamma - 1) Ma_1^2 \sin^2 \alpha}{1 + \tan^2 \alpha (\tan \beta / \tan \alpha)} \qquad (41.21)$$

Note that although $Ma_2 \sin \beta$ <1, Ma_2 might be greater than 1. So **the flow behind an oblique shock may be supersonic although the normal component of velocity is subsonic.**

In order to obtain the angle of deflection of flow passing through an oblique shock, we use the relation

$$\tan \theta = \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \cdot \tan \beta}$$
$$= \frac{\tan \alpha - (\tan \beta / \tan \alpha) \tan \alpha}{1 + \tan^2 \alpha (\tan \beta / \tan \alpha)}$$

Having substituted $(\tan \beta / \tan \alpha)$ from Eq. (41.20), we get the relation

$$\tan \theta = \frac{Ma_1^2 \sin 2\alpha - 2 \cot \alpha}{Ma_1^2 (\gamma + \cos 2\alpha) + 2} \qquad (41.22)$$

Sometimes, a design is done in such a way that an oblique shock is allowed instead of a normal shock. The losses for the case of oblique shock are much less than those of normal shock.This is the reason for making the nose angle of the fuselage of a supersonic aircraft small.

10.12 Working out solutions through Gas tables /Charts:

Not Available